Crocheting an Isomorphism between the Automorphism Groups of the Klein Quartic and Fano Plane

Shintaro Fushida-Hardy

Department of Mathematics, Stanford University; sfh@stanford.edu

Abstract

The Klein quartic is realised as a crochet model, with colours further encoding the Fano plane. The latter can be encoded in such a way that its symmetries are exactly those of the Klein quartic. This can be used to recover an exceptional isomorphism between the projective special linear groups of degree 2 over the field of 7 elements and degree 3 over the field of 2 elements.

Introduction

The *projective special linear group* of degree n over the field of q elements, denoted $\mathrm{PSL}(n,q)$, consists of certain $n \times n$ matrices whose entries are integers from 0 to $q-1$. For example, $\mathrm{PSL}(2,7)$ consists of certain 2×2 matrices with entries from 0 to 6. On the other hand $\mathrm{PSL}(3,2)$ consists of 3×3 matrices. There is no reason to expect the two groups to be related, but they are in fact isomorphic. A formal discussion of these groups and the isomorphism is provided in Wilson [5].

Such "Accidental isomorphisms", referred to as *exceptional isomorphisms*, can be difficult to visualise since they do not follow any general patterns. In this case however, $\mathrm{PSL}(2,7)$ is the automorphism group (group of symmetries) of the Klein quartic, and $\mathrm{PSL}(3,2)$ is the automorphism group of the Fano plane. To understand the isomorphism geometrically, a model was crocheted in which the symmetries of the Klein quartic and Fano plane were captured simultaneously. The Klein quartic and Fano plane are introduced in the next section, with a description of how the latter is encoded in the former. Next, the crocheted model is used to show that the symmetries of the Klein quartic and encoded Fano plane agree. An isomorphism $\mathrm{PSL}(2,7) \to \mathrm{PSL}(3,2)$ is also described using the crocheted model. Finally a description of how the model was crocheted is provided.

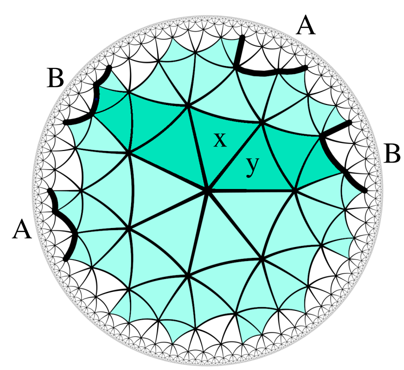

Figure 1: *The Klein quartic as a quotient of the hyperbolic plane.*

The Klein Quartic and Fano Plane

The Klein quartic can be described as a quotient of the $\{3,7\}$ triangular tiling of the hyperbolic plane [1], as shown in Figure 1. Specifically the Klein quartic can be obtained by considering the coloured triangles in the tiling and gluing the outer edges together, viz. A to A, B to B, and so on cyclically. The crocheted model follows this description; 56 triangles were crocheted and stitched together along their edges as in Figure 1.

In the introduction $\mathrm{PSL}(2,7)$ is stated as being the automorphism group of the Klein quartic. To show this, one can use the group presentation

$$\mathrm{PSL}(2,7) \cong \langle a, b \mid a^2 = b^3 = (ab)^7 = (a^{-1}b^{-1}ab)^4 = 1 \rangle$$

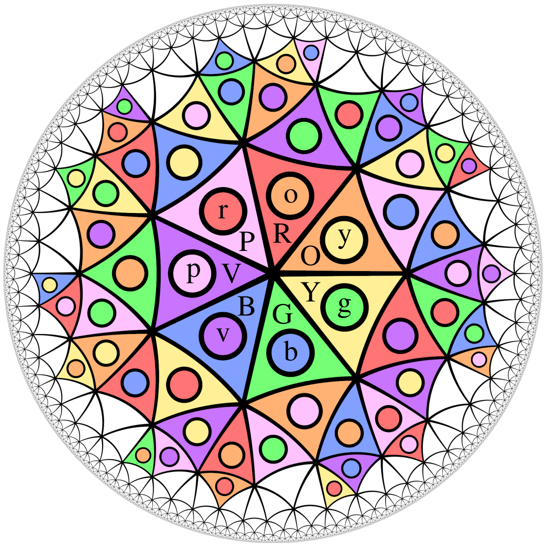

Figure 2: *The Klein quartic with colours encoding the Fano plane.*

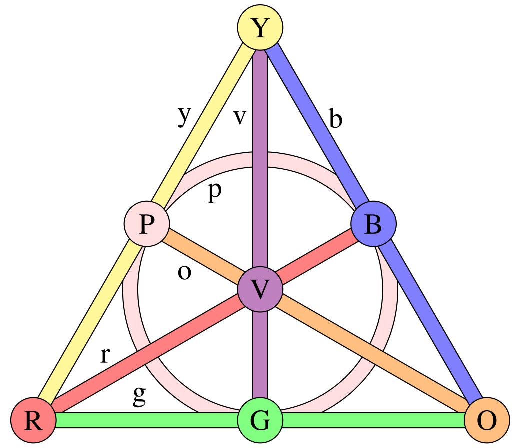

Figure 3: *The standard representation of the Fano plane as a graph.*

according to Coxeter [2]. The generators a and b can be identified with matrices $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ respectively [3]. There is no canonical action of $\mathrm{PSL}(2,7)$ on the Klein quartic, but as a specific example a can be chosen to be the π-rotation of the quartic about the edge between x and y in Figure 1 and b to be the clockwise $2\pi/3$-rotation about x. The two additional relations in the presentation are satisfied because ab corresponds to the $2\pi/7$-counter-clockwise rotation about the center of Figure 1 and $a^{-1}b^{-1}ab$ corresponds to a translation by two triangles across the shaded band. Therefore, the former has order 7 and the latter order 4. The actions of a and b can be shown to generate all of the automorphisms of the Klein quartic.

Next the Fano plane is introduced. This is encoded in the Klein quartic using colours following a similar scheme to that in Baez [1]. The Fano plane is defined as the projectivisation (space of lines through the origin) of the finite 3-dimensional vector space ${\mathbb{F}_2}^3$ [1]. Figure 3 shows the standard representation of the Fano plane as a graph in which each vertex is a point in the plane, each edge signifying that the three incident vertices are collinear. An automorphism of the Fano plane is a permutation of the vertices in which collinearity is preserved. In Figure 3, vertices have been labelled R, O, Y, G, B, V, P, each of which represents a colour: red, orange, yellow, green, blue, violet, or pink. Similarly, edges are coloured r, o, y, g, b, v, p. In Figure 2, each triangle has been given two colours, a triangle (background) colour and circle (foreground) colour. These are also labelled: R, . . ., P (backgrounds) and r, . . ., p (foregrounds).

A given point in the Fano plane (Figure 3) corresponds to the eight triangles in the Klein quartic (Figure 2) with the same background colour. Similarly, edges of the Fano plane correspond to the eight triangles in the Klein quartic with the same foreground colour. In the Klein quartic, a vertex (background colour) of the Fano plane is incident to an edge (foreground colour) if they never appear on the same triangle. The agreement of incidence relations with those in Figure 3 can be verified.

The automorphism group of the Fano plane, $\mathrm{PSL}(3,2)$, consists of 3×3 matrices. Therefore an explicit action of $\mathrm{PSL}(3,2)$ is given by matrix multiplication upon relabelling the points of the Fano plane to be vectors in ${\mathbb{F}_2}^3$:

$$\mathrm{R}=(0,0,1),\ \ \mathrm{O}=(0,1,0),\ \ \mathrm{G}=(0,1,1),\ \ \mathrm{Y}=(1,0,0),\ \ \mathrm{P}=(1,0,1),\ \ \mathrm{B}=(1,1,0),\ \ \mathrm{V}=(1,1,1).$$

The resultant labelling satisfies the collinearity relations in Figure 3.

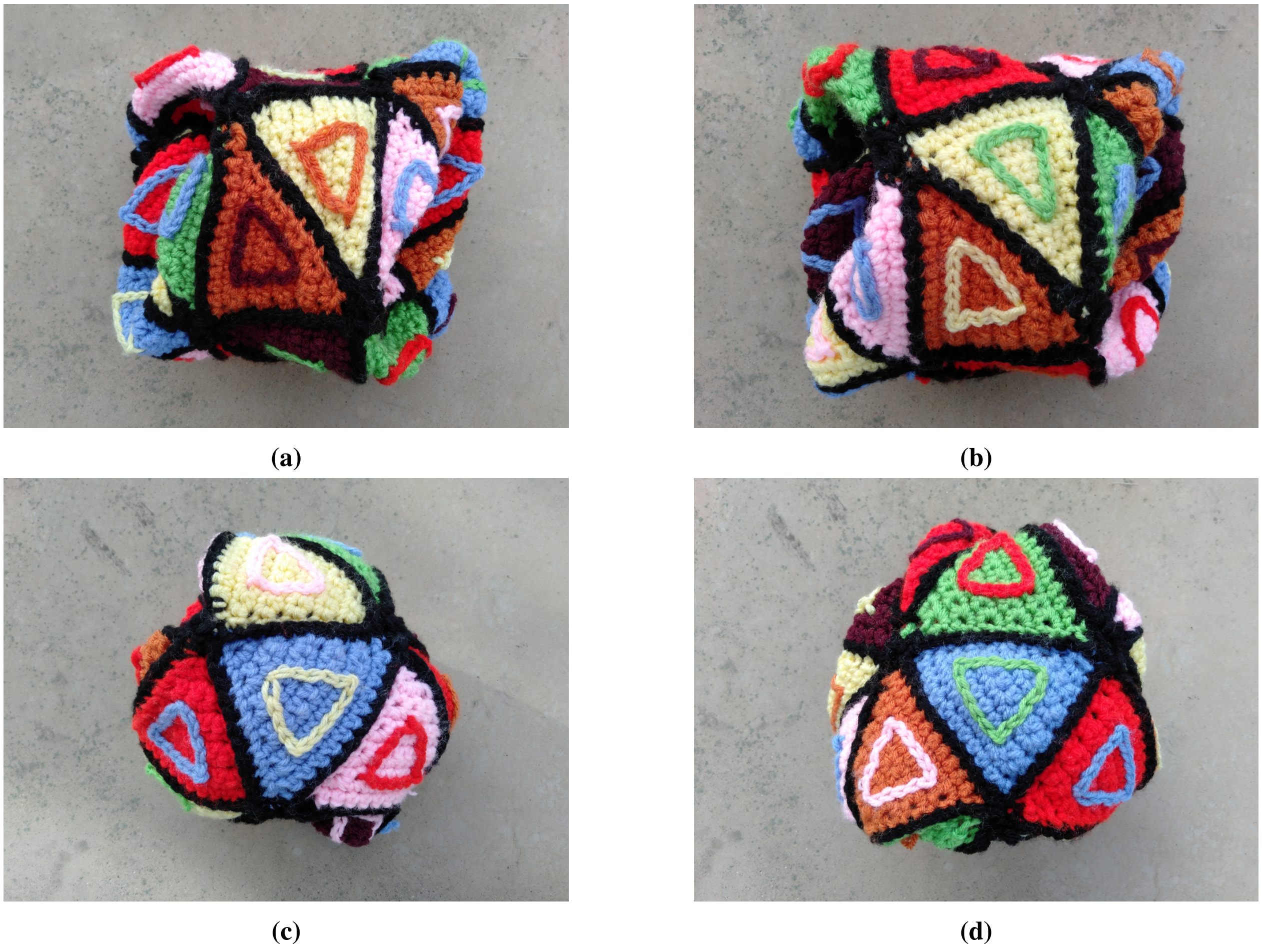

Figure 4: *Order 2 symmetries: rotating (a) or (b) by π preserves the tiling, and turning (a) upside down gives (b). Order 3 symmetries: rotating (c) or (d) by $2\pi/3$ preserves the tiling, and turning (c) through an axis in the plane of the paper by $2\pi/3$ gives (d).*

Symmetries of the Crochet

The crocheted model shown in Figure 4 uses the same colours and tiling as in Figure 2, with boundaries stitched together as in Figure 1. Use of the crocheted model to recover an isomorphism $\mathrm{PSL}(2,7) \to \mathrm{PSL}(3,2)$ requires the symmetries of the tiling (Klein quartic) to agree with the symmetries of the colours (Fano plane). This is best seen by handling the crocheted model. For example, consider Figure 4b. Rotating the figure by π is an order 2 symmetry of the underlying Klein quartic, as it preserves the tiling. On the other hand, the rotation interchanges O and P (Fano plane vertices) with Y and G, respectively, and o and g (Fano plane edges) with v and y, respectively. With reference to Figure 3, the permutation of vertices and edges preserves collinearity, so it defines an automorphism of the Fano plane. This demonstrates that the above rotation is simultaneously an automorphism of the Klein quartic and Fano plane. With the crochet in hand, one can verify that *all* symmetries are simultaneously automorphisms of the Klein quartic and Fano plane.

Matsumoto [4] demonstrated 24 of the 168 symmetries of the Klein quartic with a quilted model which used the heptagonal $\{7,3\}$-tiling. This is dual to the tiling used in the crochet model; switching every *face* in the crochet with a *vertex* and vice versa produces Matsumoto's tiling. The reader is encouraged to determine which symmetries in Matsumoto [4] are shown in Figure 4.

Like Matsumoto's quilt, the crocheted Klein quartic realised only 24 symmetries. This introduces a difficulty in describing an isomorphism $\mathrm{PSL}(2,7) \to \mathrm{PSL}(3,2)$, because not all of the matrices in these groups give symmetries of the crocheted model—some require the model to pass through itself. The missing symmetries correspond to $2\pi/7$-rotations about vertices of the tiling and their compositions with other symmetries. Fortunately in this case, the permutation of colours and hence the corresponding symmetry of the Fano plane can be easily determined by inspecting the seven triangles adjacent to the centre of the rotation. For example, choosing the vertex at the center of Figure 2 and rotating counterclockwise by $2\pi/7$ sends red to orange, orange to yellow, and so on (for both background and foreground colours). This completes the correspondence between symmetries of the Klein quartic and Fano plane.

The Isomorphism PSL(2,7) → PSL(3,2)

An isomorphism from $\mathrm{PSL}(2,7)$ to $\mathrm{PSL}(3,2)$ has been established in the previous two sections. As an example, the image of an element of $\mathrm{PSL}(2,7)$ under the isomorphism is now computed. A π-rotation in Figure 4b corresponds to a rotation about the edge between O and Y in Figure 2. In terms of Coxeter's generators of $\mathrm{PSL}(2,7)$, this rotation is given by ab^2aba, the corresponding matrix being $\left(\begin{smallmatrix} 1 & -1 \\ 2 & -1 \end{smallmatrix}\right)$. On the other hand, this symmetry of the Klein quartic crochet has been shown to correspond to the automorphism of the Fano plane interchanging O and P with Y and G, respectively. Using the relabelling of vertices as vectors, the automorphism is given by $\left(\begin{smallmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$. Therefore, the isomorphism maps the former matrix to the latter.

How was the model Crocheted?

Eight equilateral triangles were crocheted for each of seven colours. These used single-crochet stitches as they are the closest to being square. Starting with one stitch and adding an extra stitch at the start of each successive row produces an equilateral triangle. Next, the triangles were stitched together using black yarn into a large sheet as in Figure 1. Finally, the edges of the sheet were wrapped around and stitched together following the cyclic rule from Figure 1. Overall the model took about 50 hours to crochet. The main difficulty was the final step because the model required folding over itself many times and the yarn was very thick!

Conclusions and Further Crafts

The crocheted model has indeed proved useful in reconstructing an exceptional isomorphism $\mathrm{PSL}(2,7) \cong \mathrm{PSL}(3,2)$, by exhibiting the relationship between automorphisms of the Klein quartic and those of the Fano plane. There are many other exceptional isomorphisms of finite groups, such as $\mathrm{PSL}(2,4) \cong \mathrm{PSL}(2,5)$ and $\mathrm{PSL}(2,3) \cong A_4$. Further studies are encouraged to crochet geometric objects that exhibit these isomorphisms!

References

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