Infinite Quasi-Periodic Origami Tilings

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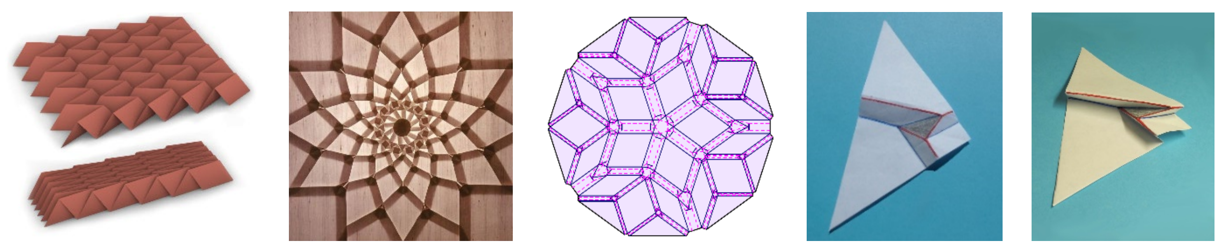
Abstract

A technique is introduced for constructing infinitely large quasi-periodic origami folding patterns. The technique builds on tiling patterns presented by Penrose and by Wang, where simple quadrilateral tiling shapes are prevented from assembling into periodic arrays by a suitable structuring or coloring of the tile edges. An equivalent edge characterization is obtained by different zig-zag folds in the tile edges.

Introduction

The Bridges audience has been introduced to origami through many intriguing finite models of cranes or other animals, and through geometric patterns such as Chris Palmer's folded curtains (Fig. 1b). In this paper we are focusing on origami patterns that could, in theory, involve the whole infinite plane. In 1970, Koryo Miura introduced an elegant periodic pattern of parallelograms that can rigidly fold in a smooth motion into an infinitely long narrow band (Fig. 1a) [6]. Some of Palmer's curtain patterns (Fig. 1b) [8] could also be extended in a symmetrical manner to cover an arbitrary large domain of the plane. But those patterns require a material that can stretch and bend in order to transform from a flat plane into the folded artistic display. Key elements in these patterns are small regions of *twist,* where a small polygonal facet is twisted to permit three or more folded bands to merge in that region (Figs. 1d, e).



(a) (b) (c) (d) (e)

Figure 1: *Origami folding of large sheets: (a) Miura Pattern [6]; (b) Palmer Curtain [8]; (c) Flagstone pattern [5]; (d, e) a triangular subdomain with a twist-region, open and folded up.*

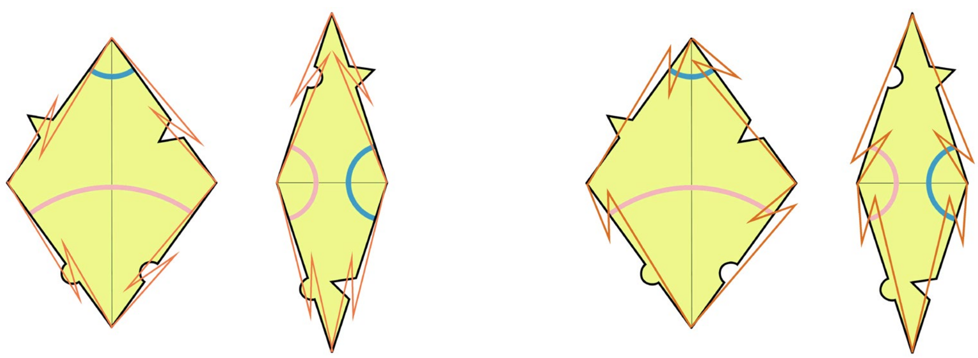
The question we asked ourselves was, whether there are similar techniques to construct origami patterns that cover the whole plane in a *quasi-periodic* manner. These are patterns where any arbitrarily large domain of the plane will not repeat itself; but significant subdomains may appear to "mostly" repeat over some finite distance. One possible approach is to start with a known quasi-periodic tiling and then replace all the tile edges by folded bands. Figure 1c shows the crease pattern for such an approach, where blue lines indicate *valley*-folds and red lines depict *mountain*-folds. When folded up, this creates a *flagstone* effect [5]. Here the initial tiling pattern is Penrose's *P3* design [7], which uses two different rhombic tile shapes with acute angles of 72 degrees and 36 degrees, respectively. To prevent these simple rhombic tiles from assembling into a regular periodic array, their edges can be modified with round and pointy serrations, or they can be enhanced with colored marks that need to match between adjacent tiles (Fig. 2). Similarly, Wang [9] has introduced an aperiodic tiling system that uses four different edge colors on eleven square tiles (Fig. 5 top). These tiles *are not allowed to be rotated or to be flipped* in the assembly process. A small portion of a resulting legal aperiodic assembly of Wang tiles is shown in Figure 5a.

Rather than beginning with a complete tiling pattern and then folding it along its edges, our approach is to design different tiles that can be joined seamlessly into infinitely large assemblies, just like the edge-decorated tiles introduced by Roger Penrose and Hao Wang. In our approach, the necessary edge characterizations are produced when folded bands run *across* the tile edges. Where a folded band runs across a tile edge, the originally straight edge will take on a *zig-zag* shape normal to the paper plane. Two tiles sharing an edge must, of course, have the same zig-zag shape, since that contact zone is part of an infinitely large, smooth sheet of "paper." The shape of that zig-zag is our means of "coloring" that edge.

In general, we partition the simple quadrilateral tile shapes into triangular subdomains, each with its own individual twist region (Fig. 1d). For a given tile, the crease pattern around the twisted area remains the same, but there is more than one way of assigning valley-folds and mountain-folds to allow complete flat folding (Fig. 1e) of that triangular subdomain. Two different assignments will result in different zig-zag folds, and this allows us to change the characterization of different tile edges.

Quasi-Periodic Folding Patterns Based on Penrose Tiling

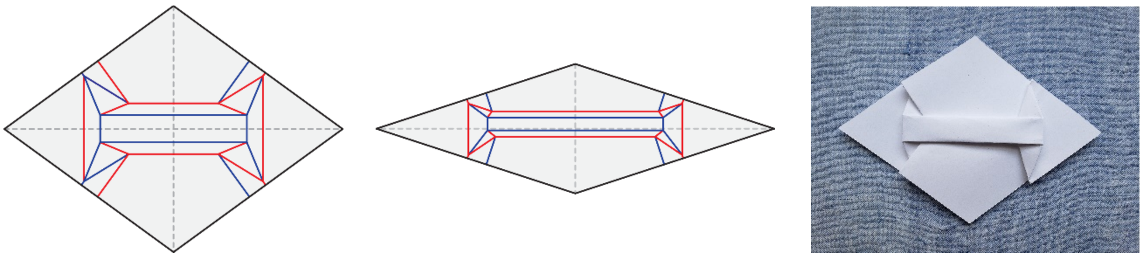
The *P3* Penrose tiling [7] needs only two different edge types; therefore, we discuss this first. Figure 2 shows the two tiles of the *P3* Penrose tiling with their edge serrations and color markings that guarantee that these tiles will combine only into a quasi-periodic pattern. The superposed orange lines show how the same constraints can be imposed by the zig-zags created where a folded band crosses the tile boundary. Just coloring the rhombus edges pink and cyan, as indicated by the ends of the circular arcs, would not be good enough, since equally colored edges with different serrations could then be joined. Correspondingly, we must make sure that the zig-zag folds do not occur in the middle of a tile's edge, but that they are properly offset towards one of its two endpoints. Depending on which way the fold has been flipped, we may encounter first a *valley* fold and then a *mountain* fold, or vice versa, as we walk along that edge in a clockwise manner around the tile center. The adjacent tile then needs to have the opposite sequence.



(a) (b)

Figure 2: *The two P3 Penrose tiles with different edge characterizations to assure quasi-periodic assemblies; (a, b) two different ways of moving the zig-zag folds off-center.*

Since a single folded band running asymmetrically through the tile edge is all that is needed, it is sufficient to partition the two rhombus shapes into just four right-angled triangles (Figs. 3a, 3b). However, constructing foldable crease patterns that contain our matching offset zig-zag conditions between tiles and subdomains of different shapes was rather challenging. We began by putting small triangular twist facets within each subdivision. They must be similar to the subdomain, but have some freedom in location, scale, and orientation. We then adjusted their position, size, and twist angles so that local foldability conditions were still met [3], and so that the tile edges fit together, and the creases making up our zig-zags remained perpendicular to the edges of the tile. It turned out that in this process, some of the small twist facets had to be moved beyond the bounds of their original subdivision domain.



(a) (b) (c)

Figure 3: *(a,b) Partitioning the Penrose tiles into four triangular domains with individual twist regions; (c) photo of a folded paper tile corresponding to the fat rhombus.*

Figure 3c shows an actual folded paper tile representing the fat rhombus. Figure 4 shows the crease pattern for a larger assembly following the *P3* Penrose tiling.

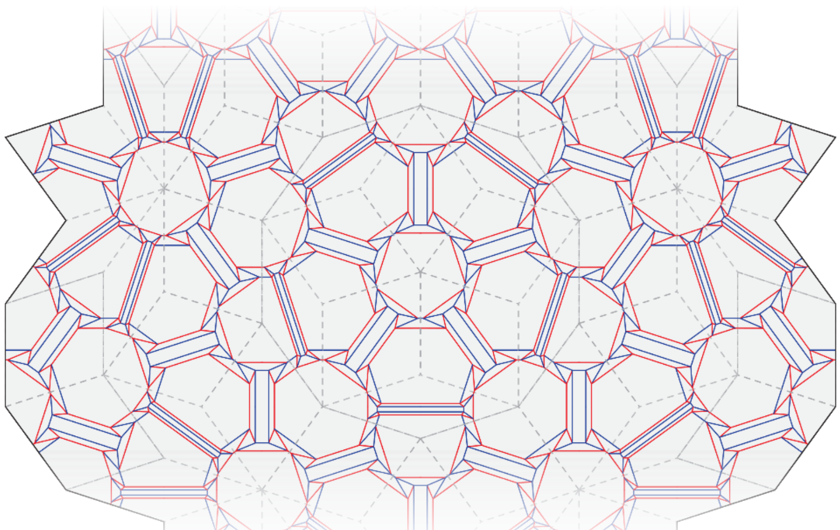
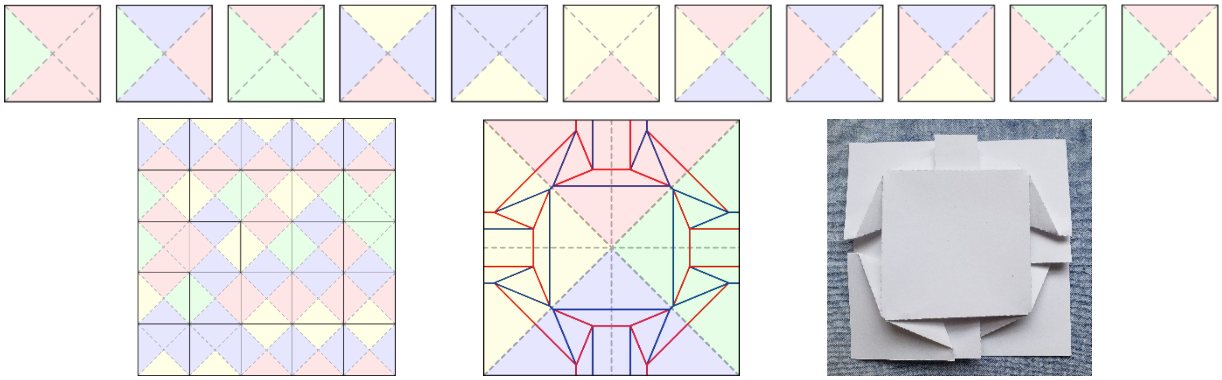


Figure 4: *A larger assembly of tiles making up the folding pattern for the Penrose P3 tiling.*

Quasi-Periodic Folding Patterns Based on Wang Tiles

The minimal Wang tiling requires 11 square tiles [4] with *four* different edge colors (Fig, 5 top). Thus we need at least "two bits" to characterize each edge. Placing an asymmetrical zig-zag on each tile edge, as we have done for the Penrose tiles, could give us the necessary two degrees of freedom. However, if we were to follow this approach, we would have to develop different crease patterns for all the eleven Wang tiles. This is a lot of work and would result in a rather irregular looking folding pattern. Thus, we route *two* folded bands cross each tile edge and thereby place *two* zig-zags in the same fixed positions on the boundaries between each tile. With this approach, the same crease pattern can be used in all eleven Wang tiles, but with different valley / mountain assignments.

There are two twist regions associated with each tile edge; and, since for each twist region there are two valid assignments for valley- and mountain-folds, we can generate four different edge characterizations with this assignment. This allows us to encode the four colors used in the chosen Wang tile set. The resulting folding pattern is very nice and symmetrical (Fig. 5b). The chosen assignments for the four differently colored tile edges show four different sequences of valley (v)-folds and mountain (m)-folds. As we move clockwise around the tile, starting at its top, we encounter the following sequences: *mvmv*, *vmmv*, *mvvm*, and *vmvm*; this corresponds to the arrangement needed for the 7^{th} tile shown at the top of Figure 5.



(a) (b) (c)

Figure 5: *(top) The 11 Wang tiles. (a) a small part of an aperiodic assembly; (b) crease pattern in one Wang tile with eight twist regions; (c) folded up Wang tile.*

Summary and Conclusions

This basic technique can be generalized to other quasi-periodic tiling patterns, where simple tile shapes are constrained by decorated edges from combining into regular periodic arrays, such as the tiles by Robert Ammann [1] or by F. P. M. Beenker [2]. However, if the tiling system requires more than "two bits" of edge characterization, or if the tile shape is not strictly convex, it becomes more difficult to break the tile into sufficiently many subdomains with their own twist regions and to route all the corresponding folded bands to the tile edges.

Beyond these tile-based approaches to construct infinitely large quasi-periodic origami patterns, there are other ways by which large aperiodic patterns could be constructed. One promising way may be to find an algorithmic process to expand one of Palmer's curtain patterns (Fig. 1b) to infinity.

Acknowledgments

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