Quasiperiodic Tilings with 12-Fold Rotational Symmetry Made of Squares, Equilateral Triangles, and Rhombi

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Abstract

We attempt to find systematically 12-fold quasiperiodic tilings using the substitution method. Only squares, equilateral triangles, and rhombi are considered as tiles. We show that a rosette with these tiles defines directly two different inflation factors. Tilings with these inflation factors are presented. We propose that many more inflation factors could be used and we show that the inflation factor determines the number of triangles used in substitution rules. Different tilings of the same inflation factor can be obtained by exchanging rhombi and squares, thereby changing the position of triangles. Imposing symmetries makes it possible to make an exhaustive search for substitution rules. Thus we obtain many new tilings.

Introduction

There exist many periodic tilings with squares and equilateral triangles, which are well known. Adding rhombi with an acute angle of 30 degrees, one gets quasiperiodic tilings of 12-fold rotational symmetry. Only some examples are shown in the tilings encyclopedia [1] and many are missing. We try to find more such tilings using a systematic approach. The results might be useful for creating new decorations.

We can easily make tiled dodecagons with 12-fold rotational symmetry. First we put together 12 rhombi and 12 triangles to get a small dodecagon with sides of the same length as the basic tiles. To this we add more tiles. If we pair triangles as rhombi we finally obtain the large dodecagon shown in Figure 1. Similar rosettes are sometimes found in ornaments, but how can we extend it to create an infinite tiling?

Substitution Method

With the substitution method we can easily create quasiperiodic tilings. It scales a tiling by an inflation factor. The enlarged tiles of the tiling are then subdivided into tiles of the original size without any gaps or overlap. Each repetition of enlargement and subdivision creates a new generation of tiles and increases the number of tiles.

We get inflation factors and substitution rules from the rosette of Figure 1. Drawing lines from its center to the middle of its sides gives an inflation factor of $2 + \sqrt{3}$ and suggests the substitution rules shown at its left. Note that this is the prevalent self-similarity ratio for tilings with 12-fold symmetry [7]. The rosette actually gives only some parts of the subdivisions. These parts are repeated symmetrically inside the inflated tiles. It is inconvenient that this substitution puts some equilateral triangles on the border of inflated tiles. Thus we use instead halves of equilateral triangles at borders, see Figure 1. This has the advantage that the overall shape of a tile does not change upon inflation and substitution. Moreover, we can thus obtain square or triangular patches of the tiling that can be repeated periodically as a decoration. From this rosette we can also get other substitution rules that give different quasiperiodic tilings with the same inflation factor. Instead of rhombi we can use squares in the substitution rule for the triangle. This gives a tiling without rosettes of rhombi and without a center of global 12-fold rotational symmetry [3]. Still it has 12-fold rotational symmetry because the tiling repeats any finite patch rotated by 30 degrees.

Furthermore, as shown in the right part of Figure 1 we can get a smaller inflation factor of $1 + \sqrt{3}$ from this rosette. But in this case only the substitution rule for the rhomb is trivially obtained and it becomes difficult to find them for the triangle and square tiles [2]. We have to choose among three different triangle substitutions to get squares and equilateral triangles across the boundaries of inflated tiles. Some questions arise: Which inflation factors can be used? How many substitution rules exist for a given inflation factor?

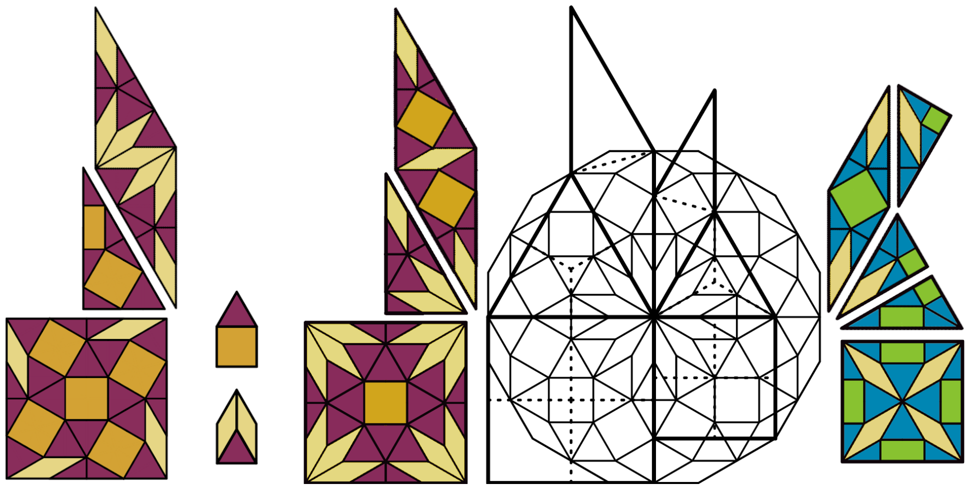

Figure 1: *At the right: Rosette of 12-fold rotational symmetry. The thick lines show inflated tiles for different ratios and the dotted lines indicate some symmetries. Substitution rules are shown at its left for an inflation ratio of $2 + \sqrt{3}$ and its right for $1 + \sqrt{3}$. Left: Modifying the tiling by replacing squares with two rhombi while shifting a triangle gives other substitution rules.*

General Observations

In 2020, Theo Schaad found other quasiperiodic tilings of 12-fold rotational symmetry with inflation factors of $3 + \sqrt{3}$ and $4 + 2\sqrt{3}$, which are yet unpublished. It is possible that such tilings can be made for any inflation factor of the form $k + m\sqrt{3}$, where k and m are integer numbers. In the substitution rules we can exchange a square by two rhombi while changing the position of a triangle, see Figure 1. Thus we can get many different substitutions for each tile. We can combine them to get a lot of new tilings.

We now show that the inflation factor determines the number of triangles in the substitution rules. For base tiles with sides of unit length the area of a square is equal to 1 and the area of a rhomb equals 1/2, whereas the area of an equilateral triangle is $\sqrt{3}/4$. The area of an inflated tile with n_s squares, n_r rhombi and n_t triangles is then equal to $n_s + n_r/2 + (\sqrt{3}/4)n_t$. For an inflation factor $k + m\sqrt{3}$, the area of an inflated square is $(k + m\sqrt{3})^2 = k^2 + 3m^2 + 2km\sqrt{3}$. From the single term with the irrational $\sqrt{3}$ we get that it has to have $n_t = 8km$ equilateral triangles such that $(\sqrt{3}/4)\, n_t = 2km\sqrt{3}$. Note that this number includes the halves at its borders. Similarly, we get $n_s + n_r/2 = k^2 + 3m^2$ for the squares and rhombi. For an inflated rhomb we get half these numbers. The area of the inflated triangle is $(\sqrt{3}/4)(k^2 + 3m^2) + (3/2)km$. This gives $n_t = k^2 + 3m^2$ for the number of its equilateral triangles and $n_s + n_r/2 = (3/2)km$. Thus for any inflation factor we know how many triangles appear in the substitution rules, and there are only a few choices for the number of squares and rhombi. Knowing in addition the tiles on the border and imposing symmetries, we can make a systematic search for all possible substitutions. Here, plastic pattern blocks are a useful tool.

Figure 2: *Tiling obtained from the substitutions at the left of Figure 1 with inflation factor of* $2 + \sqrt{3}$*. The center of 12-fold rotational symmetry lies at the lower left. Black lines show the borders of inflation and substitution steps. Initially, there is only a rosette of 12 rhombi and triangles.*

Results

Figure 2 shows the quasiperiodic tiling with an inflation ratio of $2 + \sqrt{3}$ and using the substitution rules at the left of Figure 1. It begins with the small tiled dodecagon of 12 rhombi and triangles shown in the lower left inside the small bold outline. A first iteration of inflation and substitution generates the large rosette of Figure 1 together with some surrounding tiles inside the next outline. It takes two further iterations until complete copies of the large rosette appear again, as can be seen at the right. For this inflation factor we get that an inflated triangle always has 7 small triangles. Six halves of them are already at the border, leaving 4 for the inside. Imposing 3-fold rotational symmetry we get that one triangle has to be at the center. This leaves only the two substitution rules for the triangle [3] shown in Figure 1. Proceeding similarly and imposing two-fold rotational symmetry, we get the second substition scheme for the rhomb of Figure 1. It gives a tiling without mirror symmetry [4]. For a square with 4-fold rotational symmetry we get five different substitutions, three of them do not have mirror symmetry. These substitution rules give $2 \times 2 \times 5 = 20$ different combinations. You can explore the resulting tilings with a dedicated browser app [6]. Choose the colors wisely in order to be able to see certain structures.

The tiling [2] in Figure 3 has an inflation ratio of $1 + \sqrt{3}$ and results from the substitution rules at the right of Figure 1. It begins with the same small dodecagon as in Figure 2. The first iteration gives a smaller rosette because the inflation factor is smaller. Surprisingly, the first copy of the rosette lies further away. Searching again for variations of the tiling, we get no other substitution rule for the rhomb. For the square there exists no other substitution rule with 4-fold rotational symmetry and mirror symmetry. We need three different asymmetric substitutions for the triangle halves to be able to match them to the squares and rhombi. Because of some free choices between the different triangles we get nine minor variations of the tiling [2]. Imposing less symmetry one obtains other substitution rules and tilings. An example is a tiling without mirror symmetry, having a particularly strong long range structure [5].

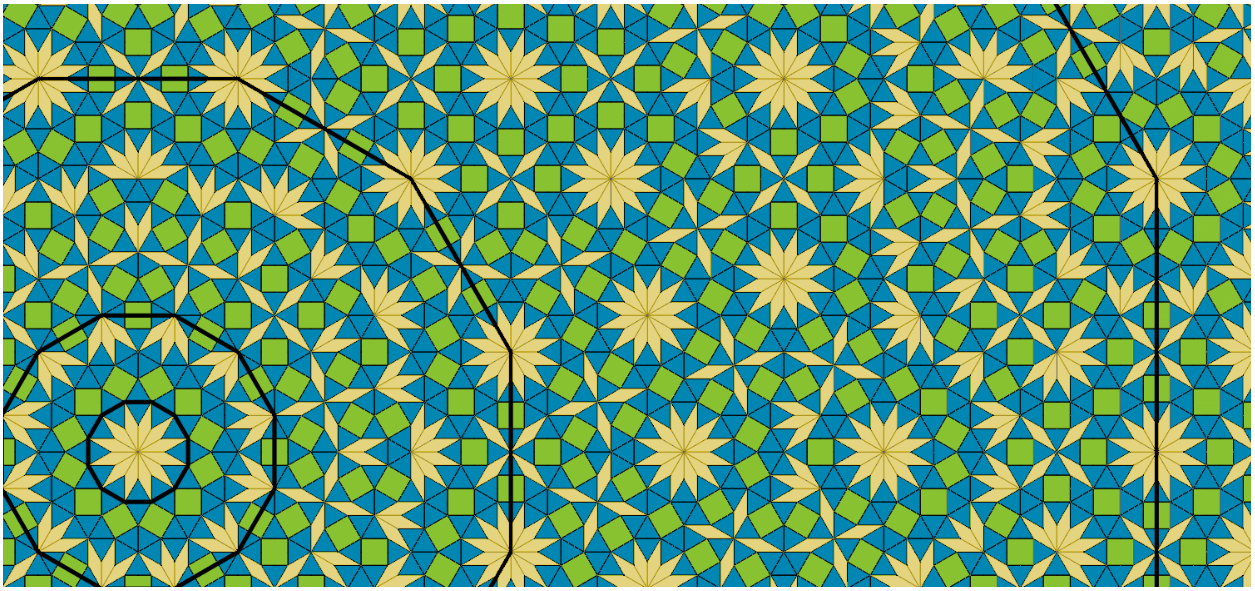

Figure 3: *Same as Figure 2, but for the right of Figure 1 and with an inflation factor of* $1 + \sqrt{3}$.

Summary and Conclusions

We have shown how to get inflation factors and substitution rules for quasiperiodic tilings of 12-fold rotational symmetry from a rather well-known dodecagonal rosette. For a factor of $2 + \sqrt{3}$ we can directly read out the substitution rules for one tiling, whereas for $1 + \sqrt{3}$ we only obtain parts of it. We propose that for any inflation factor of $k + m\sqrt{3}$ quasiperiodic tilings with dodecagonal symmetry can be found. The number of triangles in the substitution rules depend only on the inflation factor. Imposing symmetries on the tiles we can make an exhaustive search for substitution rules. Thus we find that there are many still unknown tilings for the inflation factor of $2 + \sqrt{3}$. The number of different tilings increases rapidly for larger inflation factors. Similar reasoning applies to quasiperiodic tilings of other symmetries too, in particular to eight-fold rotational symmetry.

References

[1] D. Frettlöh, E. Harriss and F. Gähler. *Tilings encyclopedia*. https://tilings.math.uni-bielefeld.de/

[2] T. P. Schaad and P. Stampfli. "A Quasiperiodic Tiling With 12-Fold Rotational Symmetry and Inflation Factor 1 + Sqrt(3)." https://arxiv.org/abs/2102.06046

[3] P. Stampfli. https://geometricolor.wordpress.com/2012/09/29/finding-an-iteration-method-for-the-stampfli-tiling-mission-impossible/

[4] P. Stampfli. https://geometricolor.wordpress.com/2012/09/30/a-tiling-with-triangles-and-rhombs-only/

[5] P. Stampfli. https://geometricolor.wordpress.com/2021/03/04/yet-another-tiling-with-12-fold-rotational-symmetry/

[6] P. Stampfli. http://geometricolor.ch/twelvefold2PlusSqrt3/twelveTwoSqrt3.html

[7] W. Steurer and S. Deloudi. *Crystallography of Quasicrystals, Concepts, Methods and Structures*. 2009, Springer, Heidelberg.