An Architectural Game of Squares and Conic Sections

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Abstract

In this paper we use the Five Points Theorem to study the possibilities of fitting a modular square grid inside a conic section.

Introduction

Originally built to be a restaurant and currently holding the university's cultural center, Casa do Lago (Lake House) is a very distinguished building in the main campus of the University of Campinas. The project by Spanish architect Joan Villà is easily recognizable by its brick walls and curved cement vaults (see Figure 1). In fact, as confirmed by [2], these features are actually an interesting attempt at bringing together Villà's own construction system based on prefabricated brick panels, Uruguayan Eladio Dieste's idea of reinforced ceramic vaults and Rodrigo Lefèvre's parabolic vaults, characteristic of the movement Arquitetura Nova (New Architecture) in Brazil.

(a)

(b)

Figure 1: *Views of Casa do Lago: (a) Entrance view of the complex (b) View of the facade of the left vault (photos by Ivan Sicca).*

This merge of techniques in the building has structural and visual consequences. One can briefly describe each of the three bodies of the building seen from the outside as a modular square structure of exposed bricks fitting under the curved vault, with the empty spaces filled with glass panels. That is also a description of the structure, since the parabolic vault is designed to hold still by itself as described in Lefèvre [3] in a way that it becomes independent of the brick structure of the facade, although they touch at the corners. It is worth mentioning that Lefèvre himself notes that the parabola is only a first approximation of a catenary, the truly self-sustaining curve, but already submitted to negligible shear stress.

Despite the references from Arquitetura Nova, as will become clearer by the end of the paper, a look at the three bottom rows of the grid which vertices are aligned along a straight line, shows that the cupola does not really satisfy this description of a parabola touching the square grid. A mathematical way to understand this is as a consequence of the known theorem that says that a conic section is completely determined by five points that it crosses. With that in mind, the goal of this paper will be to discuss what is needed for a structure to indeed satisfy our description. That is: given a conic section, which modular grids can I fit inside it? And given a grid, can I fit a conic section around it? To simplify the model and the calculations a little, we will look only at the case where the grid shares a reflection symmetry axis with the conic, as would be the case in Casa do Lago.

Geometric Model

For the model, let us assume we have squares of side k and that we know the number of squares in each row of a grid like the one in Figure 2. This translates into the parameters $x_1,\ x_2$ and x_3 in the picture being integer multiples of $k/2$ (because of that we will eventually choose $k = 2$ for some calculations later in the paper). As a clarification, notice that in the picture the conic section keeps opening up below the three rows, since $x_1,\ x_2$ and x_3 are strictly increasing. That does not have to be the case if we take $x_3 \leq x_2$, but then we cannot guarantee that the third row will be inside the curve, as the conic section will then close up over the walls in the third row.

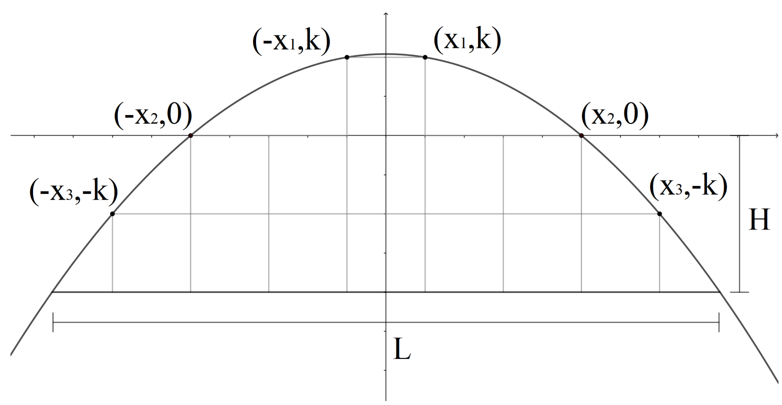


Figure 2: *General picture of the conic passing through the six vertices of the square grid. (Drawn with Geogebra 5.0).*

Application of the Five Points Theorem

There is more than one way to show that a conic section is uniquely determined by five points, here we choose to look at the generic equation of a conic section in the plane

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0. \tag{1}$$

Since the equality does not change if we multiply all the coefficients by a non-zero constant, we have five free-parameters, that can be found by solving the linear system obtained by replacing the (x, y) coordinates of five generic distinct points in the plane. To calculate the coefficients, we can arrange in the xy-plane four vertices of a trapezoid with basis parallel to the x-axis and symmetric with respect to the y-axis. That means coordinates $(\pm x_1, k)$, $(\pm x_2, 0)$ as in Figure 2.

We can assume $F = 1$, since the conic section should not cross the origin, and substitute the coordinates in equation (1). Simple calculations would then show $B = D = 0$, which means that the conic is symmetric with respect to the y-axis as expected. Including a fifth point, say $(x_3, -k)$, gives $(-x_3, -k)$ also at the conic.

Plugging this new point also in (1) we get the parameters of the conic with respect to $x_1,\ x_2,\ x_3,\ k$:

$$A = \frac{1}{x_2^2} \qquad C = \frac{2x_2^2 - x_1^2 - x_3^2}{2x_2^2 k^2} \qquad E = \frac{x_3^2 - x_1^2}{2x_2^2 k}$$

We can classify the conic section using the coefficients A, C and E. That information is given by the discriminant of the conic $d = B^2 - 4AC$, which can be positive, negative or zero, giving a hyperbola, ellipse or parabola, respectively. Since we have $B = 0$ and A strictly positive, the sign of d is the opposite of the sign of C. So we have that $C > 0$ corresponds to an ellipse, $C = 0$ to a parabola and $C < 0$ to a hyperbola.

If we need a more specific characterization of the conic section, we need the relation between other geometric quantities of the conic section and the parameters of the grid. The most important is the eccentricity e of the conic section. We can have $e = 1$, $e > 1$ or $e < 1$ corresponding, respectively, to a parabola, a hyperbola or an ellipse. As said before, if $C = 0$, $e = 1$. For the other cases, we can relate the coefficients A, C and E to e using the formulas derived in [1]. The calculations of Ayoub [1] depend on the sign of

$$\Delta = \det \begin{bmatrix} A & B/2 & D/2 \\ B/2 & C & E/2 \\ D/2 & E/2 & -F \end{bmatrix} = -A\left(\frac{E^2}{4} + C\right),$$

which is always negative in the elliptic case, but can become positive in the hyperbolic case (and even vanish for degenerate conic sections). Such sign does not depend on k. The eccentricity of the conic is then:

$$e = \sqrt{\frac{2\sqrt{(A-C)^2 + B^2}}{\pm(A+C) + \sqrt{(A-C)^2 + B^2}}} = \sqrt{\frac{2|A-C|}{\pm(A+C) + |A-C|}},$$

with the undecided sign being taken as + if $\Delta < 0$ and − if $\Delta > 0$.

Another piece of information that may be important if you are looking at a building, for example, is the span L of the conic section at a given height. Let us characterize that by the distance H between the basis of the conic section and the x-axis (see Figure 2). To do that we plug $y = -H$ in (1) with our coefficients to get

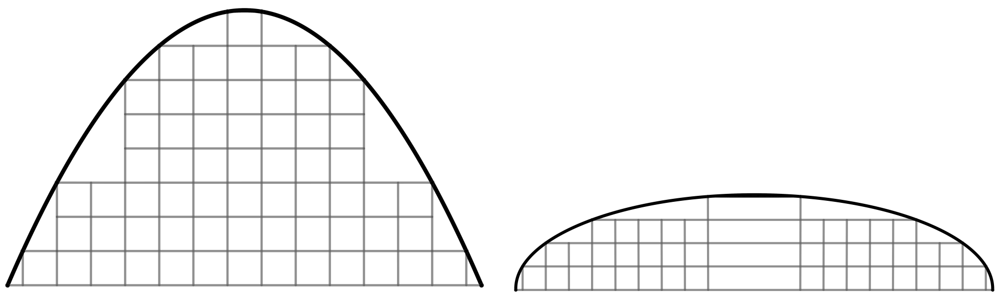
$$x = \left(\frac{L}{2}\right) = \sqrt{\frac{1 + HE - CH^2}{A}}$$

Questions and Applications

The previous section creates a single set of tools that can be used to describe or prescribe structures involving contact between modular grids and any type of conic sections you may want. The most basic question is: if I want to fit a parabola, an ellipse or a hyperbola, what numbers should I choose for x_1, x_2, x_3 and k? If we rescale the full picture by a constant α (which is equivalent to multiplying k by α in the grid), we get that our parameters transform as $C' = C/\alpha^2$, $A' = A/\alpha^2$, $E' = E/\alpha$, $e' = e$, $L' = \alpha L$ and $H' = \alpha H$. So the sign of C does not depend on k and we can assume $k = 2$ and calculate $2x_2^2 - x_1^2 - x_3^2$ for triples (x_1, x_2, x_3) of positive increasing integers, corresponding to the number of squares on each row. We can find out, for example, that $(1, 5, 7)$ would correspond to a parabola, $(4, 14, 18)$ to an ellipse (see Figure 3), etc.

Another interesting question would be to know whether or not it is possible to add more than the initial three rows of squares touching the conic. Mathematically, that translates to L being an integer multiple of k for a given H also multiple of k. Now, in the last paragraph we see that the divisibility by k is also invariant by scaling, so we can test our triplets of integers and see that, for example, the parabola for $(1, 5, 7)$ has integer values of L for $H = 8$ and $H = 12$.

If one wants to go the other way around, of filling a given conic section with a modular square grid, the basic scaling invariant information to study is e. But if we take $k = 2$ and fix x_1, x_2 and x_3 to be integers we see that only a discrete set of values for e can be attained. That means that, in general, one should not expect to be able to fit a square modular grid inside a given ellipse or hyperbola. But if you fix the x_i and allow k to change freely (allowing for grids of rectangles) you can actually adjust the parameters to get a desired eccentricity.



(a) (b)

Figure 3: *Some examples of grids matching conic sections for given triplets* (x_1, x_2, x_3) *(a) (1,5,7) gives a parabola.*
(b) (4,14,18) gives an ellipse which is one of the few with an integer value of L in the forth row (images generated with the software Geogebra).

Summary and Conclusions

As mentioned before, applying this method to study Casa do Lago one finds out that the calculations disagree with the model of a parabola with a touching square grid, and a deeper analysis of the building should be made to describe it properly. So the method does have a place in the study of the precise shapes of some constructions, even if to give a negative answer like here.

As a side note, when doing pictures like the ones in this paper, thanks to the tool "Conic through five points" in Geogebra, one can anchor the full construction on the choice of the first three points and obtains a sort of graphical calculation of all the information one may want for the conic section, like L for a given H or the total height of the conic section. This might be useful not only for artists or architects exploring with this forms but also may constitute an interesting exercise for students of any course that may cover these curves.

Acknowledgements

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References

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