

Sculpture Design with Hexastix and Related Non-Intersecting Cylinder Packings

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Abstract

We explore periodic, non-intersecting cylinder packing arrangements known as hexastix and polystix. Groups of congruent cylinder arrangements, related through cubic symmetry, and limited to only three or four directions are defined and classified in this paper. The basic techniques and design principles required to reproduce several polystix models are illustrated. Adding a layer of complexity, we look at the different ways the ends of these symmetric stick weaving constructions can be joined to create intricate knots and links. Several original artworks made with pencils or flameworked glass, based on these cylinder packing arrangements, are presented and explained. Building and playing with these structures is a fun way to explore symmetry and promote mathematics as beautiful art.

Introduction

Polystix are an interesting family of infinitely-repeating, symmetric rod arrangements with rods parallel to only 3 or 4 directions. I started playing with these structures after seeing a sculpture, titled “72 pencils” [5], made by the prominent math artist, George Hart. A quick internet search led me to Alejandro Eriksson’s website [4] that contained detailed and easy-to-follow instructions for making hexastix. I was intrigued to find the exacting symmetry could be made without using complicated templates or specialized fixtures. Pencils are held in place with the tension from rubber bands, organizing an otherwise random pile of sticks into a surprisingly rigid and beautiful geometric sculpture. After making a few, I could see the hexagon was just one possible subset lying on a larger, infinite grid. The same basic techniques could be used with other shapes and patterns to create novel structures. I use these cylinder packings as the bases for making glass sculptures. These sculptures explore how the ends of the rods can be joined together to create symmetric knots. Specific background is presented first, in order to explain the structure of the original artworks.

Classification

The geometry of non-intersecting rod structures has been used for hundreds of years, with uses found in interlocking burr puzzles, architecture, and art. The underlying mathematics has only started to be addressed and classified over the past 50 years. There are several instances of cylinder packings being independently discovered. Although they have not received as much attention as sphere packings, they are also useful in aiding our understanding of how space is organized.

Two well-known, early examples of these forms can be seen in *Shapes, Space and Symmetry* [6], where Holden illustrates ways to fill the holes that are left when rhombic dodecahedra are organized into a vertex-touching arrangement, along with a similar arrangement made from cubes. In *The Symmetries of Things* [2], Conway, Burgiel, and Goodman-Strauss describe and name these arrangements in a section addressing objects with prime symmetry. They classify this family of infinite packings within cubic space groups and give them the name “polystix.” Sticks with a square cross section, in a simple primitive cubic non-intersecting arrangement, are labeled as “tetrastix.” The alternative cubic packing is named “hemistix.” A chiral arrangement of triangle sticks in 4 directions is known as “tristix,” and hexagonal sticks in 4 directions are “hexastix.”

Similarly, infinite cylinder packings are described comprehensively and from a crystallographic point of view, by Michael O’Keeffe, Regents Professor Emeritus ASU, and associates, in their papers [8,9,10]. O’Keeffe describes repeating rod structures with crystallographic symmetry operations and also defines their cubic space groups, determining that there are only 6 possible invariant rod packings of 3 and 4 directions. They additionally describe possible homogeneous packings with rods in 6, 12, and 24 directions, and introduce the idea of replacing single cylinders with bundles of cylinders. Packings are listed in descending density, with the homogeneous packings numbered and the 6 invariant packings labeled with capital Greek letters. It is stated that the hexastix, or “T” packing, is the densest possible cubic packing. However, when using the optimal polygonal cross section of rods, both hexastix and tetrastix occupy exactly 75% of space.

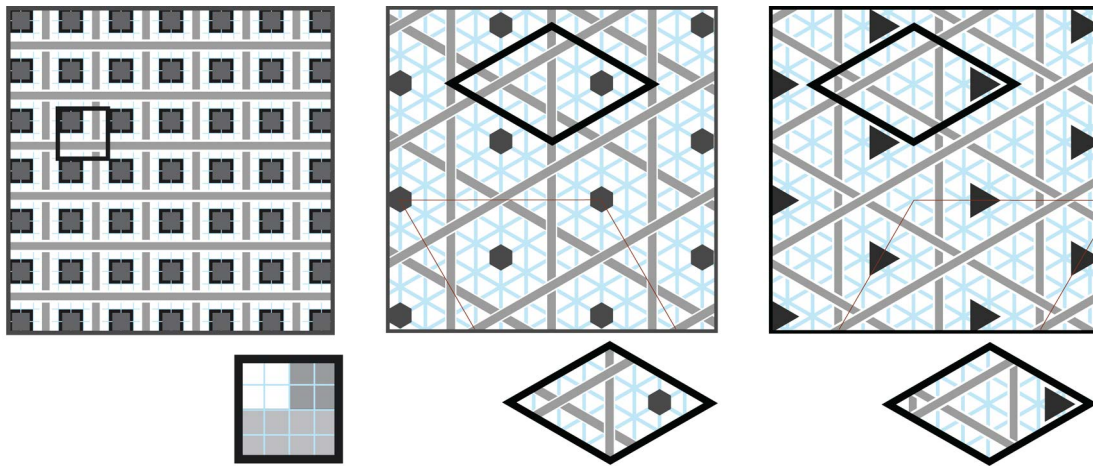


Figure 1: *The 3 polystix grids with unit cells outlined.*

Infinite packings have every cylinder in contact and held immobile by all of their neighbors. Models made with congruent cylinders of finite length do not have infinite neighbors restricting each other in place. All configurations of straight rods can come apart and are not mechanically stable without being held in place with glue or being fixed by some other means. The simplest way to keep polystix models from coming apart is to use compression. String or rubber bands can be used to bind the ends of rods (that are on the same axis) together to create structural rigidity through friction and compression. Puzzlers resolve the problem of keeping unstable arrangements from coming apart by introducing different kinds of interlocking mechanisms. Interlocking burr puzzles have components added or subtracted to hold or lock the other pieces in place. These burr puzzles present their own set of combinatoric and geometric problems, many of which have been addressed extensively by Cutler [3] and Coffin [1]. John and Jane Kostick have also discovered and developed many of these and other symmetric star arrangements, realized through a wide array of impressive sculptures and puzzles [7]. I have yet to uncover a complete record or documented history of polystix structures, the examples noted above are just a few I find to be especially innovative, comprehensive, and beautiful.

Modeling Polystix

Polystix designs can be simplified and explored in 2-dimensions, as all of the homogeneous arrangements of 3 and 4 directions use only one of 3 grids, shown in Figure 1. Using the compression from rubber bands, a variety of simple materials such as; toothpicks, skewers, pencils, or any uniform rod or stick can be organized into these geometric forms. The tetrastix 3-way arrangements are simple to construct and a good introduction for beginners. Hexastix is more complex, but only slightly more difficult to make, as it can be

constructed by penetrating one of the diagonal axes of a tetrastix arrangement. Hexagonal pencils work perfectly for constructing hexastix. Although hexagonal pencils are not the optimal shape for constructing tetrastix and tristix, they can be used to create these arrangements. Hexagons and other patterns can be worked out beforehand on 2-dimensional grids, as they should repeat on each axis, as in Figure 2.

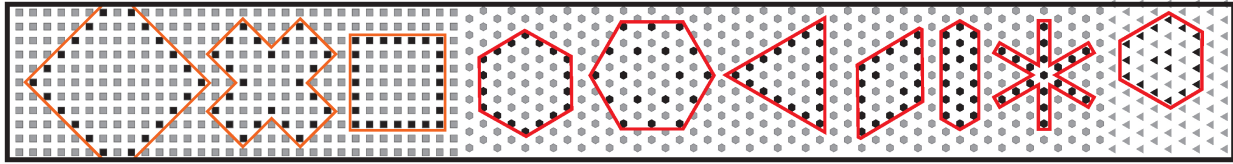


Figure 2: 2-dimensional patterns used in “10 Arrangements of 72 Pencils”. Tetrastix, hexastix, and tristix on their respective underlying square, hexagonal and, triangular grids.

The hexagonal pattern used in “72 pencils” (4th from left in Figure 2) has a nice recursive quality of being the shape of the rods themselves. Triangles, crosses, hearts, and many simple shapes can be outlined and used to create unique structures. The work I made for the JMM 2020 art exhibition [11] (Figure 3) shows 10 different arrangements, that have the number of rods used in each piece restricted to 72, based on the original “72 pencils” sculpture.

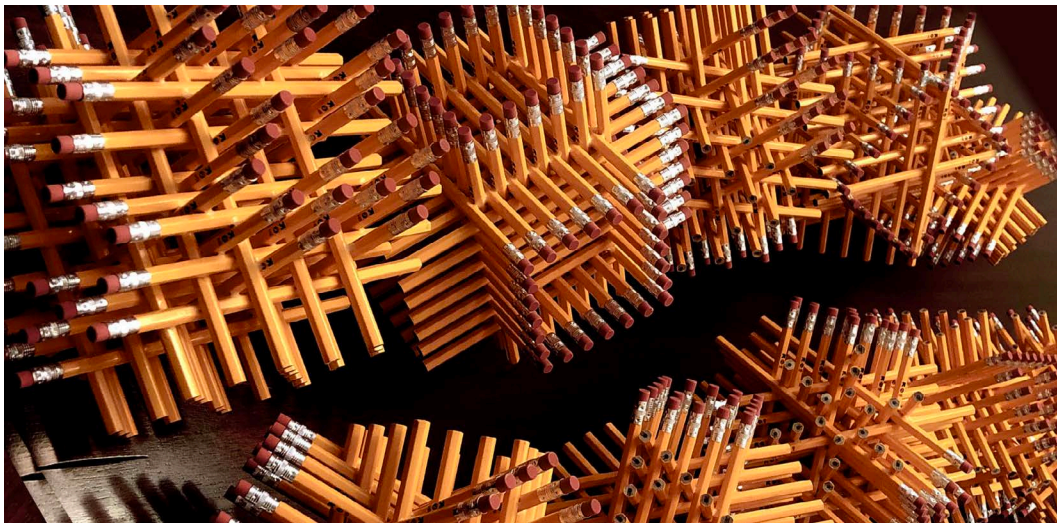


Figure 3: 720 pencils, “10 Arrangements of 72 Pencils”.

Units can be joined like crystal building blocks or polyhedral bricks to make larger forms with short rod segments. An infinite array of polyhedra, chains, spirals, and other forms can be made by adding small units together. Modeling polystix is an engaging, hands-on activity that provides a wide range of design choices including patterning, coloration and rod shape, that allow for a great deal of creativity and play.

Constructing Knots

Polystix are used in my work to design more complicated glass sculptures, joining the ends together in various ways to create knots and links, such as Figure 4 and [12]. When looking at possible ways to connect the polystix arrangements, I started with six rods in a tetrastix/Borromean arrangement and found there are 3,840 unique paths or ways to connect the ends into a continuous loop. With so many end-connecting possibilities, I focused on reducing the number of potential links by only using the neighboring connections that make an embedded loop. The knots are constructed by using either “twists” to join proximal neighbors that cross on different axes or “half loops” to join neighbors on the same axis.

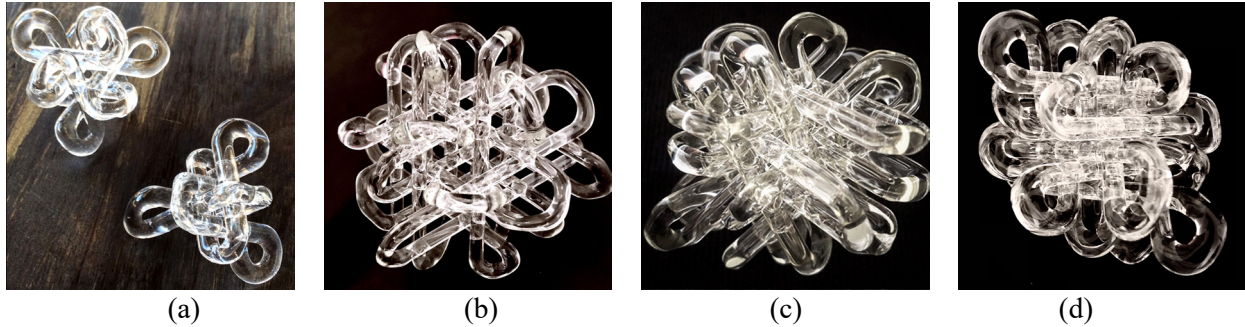


Figure 4: Knots made from cylinder packing arrangements: (a) Two tetrastix knots, (b) Alternate hexastix tetrahedral 4 twist, (c) Hexastix tetrahedral 4 twist, (d) Hexastix 12 twist.

Using this neighbor-linking system, a single arrangement can be connected in many different ways to make a variety of unique knots, e.g., Figure 4a. Creating polystix sculptures in glass presents some unique technical challenges. Glass rods are initially held in place using steel wire. By using a propane and oxygen torch, I carefully melt, bend, and join the ends of borosilicate glass rods to create uniform transitions. Constructing polystix with glass allows light to bounce in and around the interior of the knots, highlighting and accentuating the beauty within the woven spaces.

Conclusions

Building models is a fun, hands-on way to encourage mathematical thinking and build spatial awareness. The various reusable and low-cost materials that can model polystix makes it a very sensible tool for exploring packing problems, as well as for different kinds of puzzle design and sculpture. The methods and examples illustrated in this paper can be used to design a large variety of unique geometric forms and can serve as an introduction to more complicated weavings, entanglements, and molecular knot structures. Polystix are versatile arrangements with many design possibilities that are yet to be explored. Upcoming workshops, lesson-plans, and sculptures [13] will continue to investigate and expand on cylinder packing ideas. My hope is that polystix arrangements encourage others to use mathematics as a way to play with art, or use art as a way to engage with mathematics.

References

- [1] S. Coffin. *Geometric Puzzle Design*, A K Peters Ltd, Massachusetts, 2007.
- [2] J. Conway, H. Burguel, C. Goodman-Strauss. *The Symmetries of Things*, CRC Press, Boca Raton, FL, 2008.
- [3] W. Cutler. “The six-PieceBurr.” *Journal of Recreational Mathematics*, 10 (4), 1978, 241–250.
- [4] A. Erickson. 2021. <http://alejandroerickson.com/j/2011/09/27/090-how-to-make-hexastix-detailed-photo-instructions.html>
- [5] G. Hart. *72 Pencils*. <https://www.georgehart.com/sculpture/pencils.html>
- [6] A. Holden. *Shapes, Space and Symmetry*, Columbia University Press, New York, 1971.
- [7] Jane and John Kostick. 2021. www.kosticks.com/stars.html
- [8] M. O’Keeffe and S. Anderson. “Rod Packings and Crystal Chemistry.” *Acta Cryst.* A33, 1977, 914–923.
- [9] M. O’Keeffe, J. Plevert, Y. Teshima, Y. Watanabe, and T. Ogama. “The Invariant Cubic Rod (Cylinder) Packings.” *Acta Cryst.* A57, 2001, 110–111.
- [10] M. O’Keeffe, J. Plevert, and T. Ogama. “Homogeneous Cubic Cylinder Packing Revisited.” *Acta Cryst.* A58, 2002, 125–132.
- [11] A. Widmark. “Stixhexaknot: a Symmetric Arrangement of Knotted Glass” *Journal of Mathematics and the Arts*, vol. 14, no. 1, 2020, 167–169.
- [12] A. Widmark. “Stixahexaknot” and “10 Arrangements of 72 Pencils.” 2020 Joint Mathematics Meetings Art Exhibition Catalog, <http://gallery.bridgesmathart.org/exhibitions/2020-joint-mathematics-meetings>
- [13] A. Widmark. 2021. www.andurielstudios.com