Variations of the Goldberg Ground and Other Canonic Adventures

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Abstract

The first four of J. S. Bach's Fourteen Canons on the Goldberg Ground are more mathematical than musical. Rather than being developed pieces, they are written in the form of puzzles that instruct the reader in basic techniques used to form canons. I propose mathematical notation for representing two-part canons and use this notation to explore the first four Goldberg ground canons. My methods are independent of European common-practice music theory and can be used to create new canons of melodies, rhythms, or other features in different musical styles.

Canons on the Goldberg Ground

Bach's Fourteen Canons on the Goldberg Ground [1] are short canons that were discovered in 1974, handwritten by Bach in his personal copy of his Goldberg Variations (1741). The Goldberg ground, labeled "1." in Figure 1, is not his own composition; it is an eight-note motif that appears in the bass line of many Baroque pieces, including the Goldberg Variations [5]. Each of the fourteen canons involves playing the Goldberg ground together with one or more transformations of itself. Bach cleverly hid the instructions for transforming the ground in the notation of the canons, called "puzzle canons" for this reason [3]. For example, the reversed bass clef in Canon #1 tells us to play the ground in reverse. YouTube user gerubach uses computer animation to demonstrate the solution to these puzzles visually [4].

Wolff [8] describes Bach's Fourteen Canons on the Goldberg Ground as "theoretical" because, in an almost mathematical way, they illustrate basic techniques used in forming canons, rather than presenting as developed musical pieces. Haakenson [5] applies European common-practice musical theory to describe the basic structure of the two-part Canons #1-4, uncover some special properties of the Goldberg ground, and compose new canons that have these properties. More broadly, two-part music exists in many, if not most, musical cultures around the world. Using mathematical notation, I abstract the structure of Canons #1-4 and suggest how we may form new canons of melodies, rhythms, or other features.

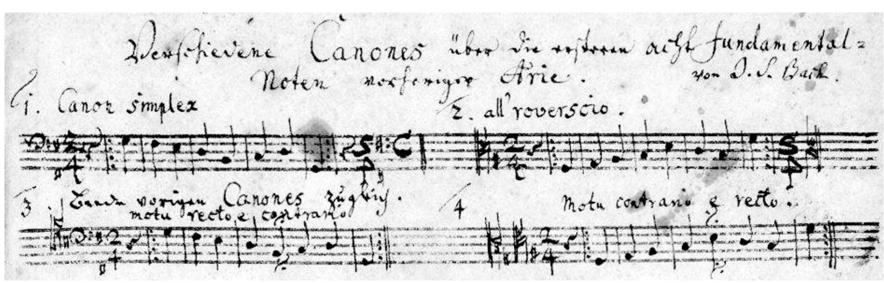

Figure 1: *Canons #1-4 of J. S. Bach's Fourteen Canons on the Goldberg Ground (BWV 1087). Image: Public domain, via Wikimedia Commons.*

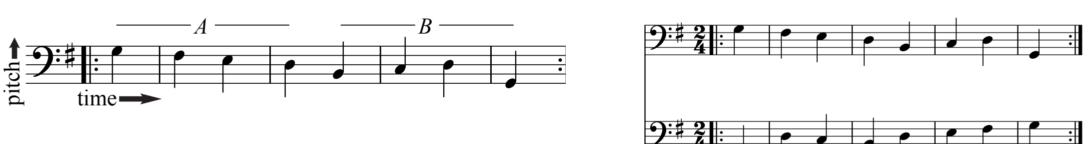

Figure 2: *(left) The Goldberg Ground; (right) Canon #1.*

Affinity and Transformation

When two sequences of sounds, *X* and *Y,* produce a succession of "pleasing" combinations when played simultaneously, whatever "pleasing" means in the musical context, let's write *X* ♥ *Y* (why not!) and say that *X* and *Y* "have an affinity." Note the difference between "a succession of pleasing sounds" and "a pleasing succession of sounds"—*X* ♥ *Y* refers to conditions at points in time, rather than an overall suitability. The affinity relation is symmetric (*X* ♥ *Y* iff *Y* ♥ *X*) but is not assumed to be transitive or even reflexive. That is, if *X* has an affinity for both *Y* and *Z*, it is not guaranteed that *Y* ♥ *Z*; moreover, *X* ♥ *X* is not guaranteed.

In Western-style counterpoint, we might think of *X* a melody, Y as a countermelody, and *X* ♥ *Y* to mean that the notes of *X* and *Y* form consonant intervals at all times. However, one should not assume that the affinity relationship applies only to Western harmony. Affinity relationships between two simultaneous sounds occur in the domains of pitch, rhythm, timbre, etc., and in many styles of music. Conversely, in Western-style counterpoint, the fact that the notes of *X* and *Y* form consonant intervals is not enough to satisfy the rules of counterpoint, which stipulate, among other things, whether two consonances may follow one another and how pieces end and begin, and which, in fact, sometimes permit dissonant intervals.

Musical transformations are certain invertible functions that map one sequence of sounds to another. Two commonly used transformations are retrograde, *R*, which reverses the order of sounds, and inversion, *I*, which interchanges low and high pitches of a pitched sound. While "retrograde" is defined for any sequence of sounds, "inversion" is only used in pitched music. There are additional transformations that apply in other contexts. Since *R*(*X*) and *R*(*Y*) are *X* and *Y* played backwards, *X* ♥ *Y* iff *R*(*X*) ♥ *R*(*Y*), and, for any fixed permutation of a sequence of sounds *P*, *P*(X) ♥ *P*(*Y*).

In Canons #1-4, at every point in time, the harmonic "slice" consisting of two notes sounded together forms a consonant interval: in the context of Baroque two-part counterpoint, the pitches may differ, modulo 7, by zero, two, four, or five scale steps, but not by one, three, or six steps. In this context, "inversion" means "scalar inversion," which preserves the number of scale steps while reversing high and low notes. Thus, if *X* ♥ *Y*, then *I*(*X*) ♥ *I*(*Y*).

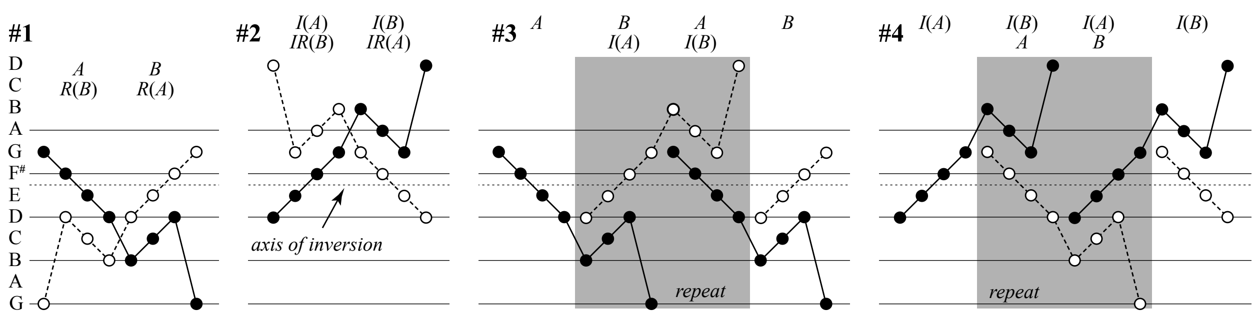

Figure 3: *Graphical Representation of Canons #1-4 from Fourteen Canons on the Goldberg Ground.*

Analysis of Bach's Canons #1-4

A crab canon occurs when a sequence of sounds is played simultaneously with its retrograde. To form a crab canon, we must find *X* so that *X* ♥ *R*(*X*). If we divide *X* into two equal-length fragments *A* and *B,* as shown for the Goldberg ground in Figure 2, left, we write *X* = [*A B*] and require that [*A B*] ♥ [*R*(*B*) *R*(*A*)]. Since *A* is played simultaneously with *R*(*B*) and *B* with *R*(*A*), *A* ♥ *R*(*B*) and *R*(*A*) ♥ *B*. An efficient way to construct a crab canon is to find a sequence *C* so that *A* ♥ *C*, then set *B* = *R*(*C*). Using this technique, YouTube user dananddanfilms produced a wonderful play that is also a crab canon [2].

Canon #2 features the inversions of the Goldberg ground and its retrograde. As in a geometrical reflection, the axis of inversion must be specified. In Bach's first four canons, "inversion" corresponds to the horizontal reflection that exchanges E and F#, D and G, C and A, and so on, as shown in Figure 3. Note that $RI(X) = IR(X)$ for any inversion of a melody X. In Canon #2, the inversion and retrograde inversion of the Goldberg ground are played together; $I(A)$ ♥ $IR(B)$ and $IR(A)$ ♥ $I(B)$.

Canons #3 and #4 employ inversion but not retrograde. However, the two players do not start at the same time. In Canon #3, the first player repeats the ground. The second player starts after four notes and plays the inversion of the ground. These instructions are indicated by the segno showing where the second player enters and the position of the double clefs. Both players repeat their melodies several times. The canon ends when the first player completes the ground and the second player finishes halfway through its inversion. In notation, the first player repeats [A B] several times and the second player repeats [$I(A)$ $I(B)$]. During the repeated section, A ♥ $I(B)$ and $I(A)$ ♥ B.

Oddly, Canon #4 introduces no additional information about affinity. The difference between Canons #3 and #4 are in what is heard *before and after* the repeated section. We might expect "Canon #3a" and "Canon #4a" which are the retrogrades of Canons #3 and #4 in the repeated section. However, stipulating that both players end at the time where the first player's melody ends would mean that the canon ends with the first player on G and the second on B (in #3a) and the first player on D and the second on B (in #4a). Both of these intervals are so-called "imperfect" consonances, and, presumably, Bach wanted to end with a perfect octave.

What is needed to write a variation of the Goldberg ground—a melody that can form canons in these four ways? We require A ♥ $R(B)$, A ♥ $I(B)$, $R(A)$ ♥ B, and $I(A)$ ♥ B. As previously stated, A ♥ $R(B)$ iff $R(A)$ ♥ B. Additionally, $I(A)$ ♥ B guarantees that A ♥ $R(B)$, and the problem reduces to finding A and B so that $R(A)$ ♥ B and $I(A)$ ♥ B. However, Bach employed a neat feature of the Goldberg ground, as observed by Haakenson [5]. Because of the particular axis of inversion chosen and the symmetry of A, $A = RI(A)$. Therefore, $R(A)$ ♥ B implies $R(RI(A))$ ♥ B, so $I(A)$ ♥ B. In addition to the stipulations (1) that A be chosen to satisfy $A = RI(A)$ (or B satisfies $RI(B) = B$), which occurs if the sequence of intervals between the notes of A is palindromic and the axis of inversion bisects the middle interval, and (2) that $R(A)$ ♥ B, Haakenson states conditions that ensure that the resulting canons satisfy *all* the conventions of Baroque music, rather than just being a sequence of consonant intervals.

Further Canonic Adventures

None of Bach's Fourteen Canons are rounds, where both parts play the same melody starting at different times. Why not? Figure 4 shows the result of performing the Goldberg ground as a two-part round. The highlighted intervals—an augmented fourth and a major second—are dissonant; even if dissonances are allowed sparingly, it would not do to have two in a row. However, since we know that A and $R(B)$ harmonize, we could write a new melody, [A $R(B)$], formed from A followed by the retrograde of B. This melody works nicely as a round, as shown in Figure 4. The reader is encouraged to discover other such "Goldberg Rounds."

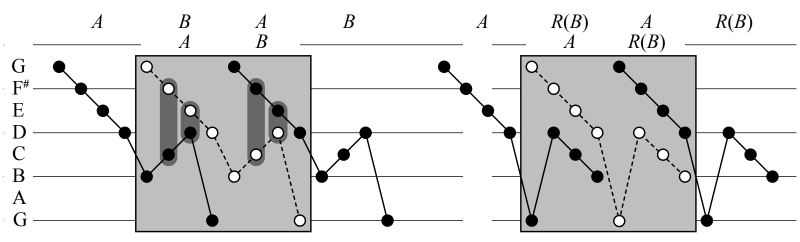

Figure 4: *The Goldberg ground as a round (left, with dissonances highlighted) and a variation of the ground that works well as a round (right)*

Rhythmic canons are canons in which rhythms, and not necessarily melodies, are duplicated by each voice. Drumming patterns formed from a constant beat may be written as a succession of `x`'s, indicating drum hits, and `–`'s, indicating rests. A rhythmic canon is complementary if, on each beat, no more than one voice has a drum hit. A tiling canon is a complementary canon that has exactly one drum hit (in some voice) per beat. See [6] for more on rhythmic canons.

The complement transformation, C, exchanges hits and rests. If there are only two players, an affinity relationship is $X \heartsuit Y$ iff $Y = C(X)$. This is equivalent to the "tiling" condition that we have exactly one drum hit on each beat when X and Y are played together and is essential to playing instruments like the panpipes or handbells. A two-part round is produced when the first player plays pattern A followed by the complement of A, $C(A)$; the second player starts playing A when the first player plays $C(A)$. For example, if A = `x-xx`, then the first player plays $[A\ C(A)]$, or `x-xx-x--`, and the second player starts after four beats with the same pattern. Alternately, the canon could be performed by one person using right and left hands, producing `RLRRLRLL`, a pattern well known to drummers as a "paradiddle." Playing any pattern with a right-hand lead followed by the same pattern with a left-hand lead produces a "round" between the right and left hands.

A tiling crab canon is formed from a rhythm sequence A by following A with the retrograde of its complement. For example, setting A = `x-xx` gives a tiling canon where the first player plays $[A\ RC(A)]$ = `x-xx--x-` and the second plays its retrograde, $[R(R(C(A))\ R(A)] = [C(A)\ R(A)]$ = `-x--xx-x`. Combining the rhythms as right and left hand gives us the pattern `RLRRLLRL`.

Could a rhythm pattern form two different styles of canons, as the Goldberg ground does? If and only if A is a palindrome, like `x--x`, then $[A\ C(A)]$ (here, `x--x-xx-`) can produce both a crab canon and a round. This is an unsatisfactory answer, as the two canons are identical! We could alter the definition of "affinity" to mean that $X \heartsuit Y$ iff Y is complementary to X. In this case, B must be complementary to both A and $R(A)$. Using A as the famous clave [7], `x--x--x---x-x---`, and B as `-x--x--x---x-x--` gives Figure 5.

```
|x--x--x---x-x---|-x--x--x---x-x--|      x--x--x---x-x---|:-x--x--x---x-x--|x--x--x---x-x---:|
|--x-x---x--x--x-|---x-x---x--x--x|                      |:x--x--x---x-x---|-x--x--x---x-x--:|

|R-LRL-R-L-RLR-L-|-R-LRL-R-L-RLR-L|      R--R--R---R-R---|:LR-LR-LR--LRLR--|RL-RL-RL--RLRL--:|
```

Figure 5: *A crab canon (left) and round (right) formed from the clave. The third line shows the canons' interpretation as drum patterns for right and left hands.*

Summary and Conclusions

This short paper has only scratched the surface of the formation of two-part canons. Readers are encouraged to compose canons themselves! In addition, this mathematical view of canons may reveal connections between canonic techniques that are already in use in various kinds of music around the world.

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