The Tower of Ha(rmo)noi

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Abstract

By use of sonification (the analogue of visualization, but with sound) based on familiar patterns of musical harmony, I set up a scheme to associate chord sequences with the steps in solving the mathematical puzzle known as the Tower of Hanoi. The aim is to obtain sonifications such that the harmonies generated produce a musical resolution coincident with the solution of the puzzle, and I present some examples in this vein. Such sonifications are of course not unique, and the intention is to select choices that are aesthetically pleasing while mapping information in a meaningful way. I close with some thoughts about the application of such methods to other mathematical puzzles, including the possibility of turning this relationship on its head, and using the solving of puzzles as a tool for composing music.

Introduction

Visualization has become an important area for representing the properties of mathematical information [4]. The original motivation behind data visualization, for example, is often to find a means to make the data more readily comprehensible [11], but, in so doing, it is often the case that the resulting product is something that can be appreciated artistically [1]. For example, many of the false color images of deep space objects obtained using observations by the Hubble space telescope [5] satisfy both these desiderata [10].

While it is less widely used, one can also represent mathematical information auditorily, creating an analog of visualization known as sonification [7]. There are various reasons one might wish to represent data with sounds. There might be situations in which a person's vision is already occupied with another task, or other situations in which the ability of sound to travel without line of sight restrictions (due, fundamentally, to the longer wavelengths and greater diffraction of audible sounds compared to visible light) might be advantageous. Another potential benefit of sonification is that our ears perform Fourier analysis on sounds so we can hear multiple notes at once, whereas our visual system blends multiple inputs into a single perceived color (e.g., red and green pixels combine to make yellow, but the notes C and E do not combine to make D).

One feature of sound is that if one has experience with a musical tradition, sonification can take advantage of people's expectations based on that tradition. In this paper, I describe work that does just that. I present an approach to the sonification of the Tower of Hanoi that generates a sequence of chords based on the sequence of intermediate states of the puzzle. The idea is to assign sounds in a way that not only is a map from those intermediate states, but provides a harmonic resolution within the framework of harmonies used in classical Western music when the puzzle is solved. The emphasis on harmonic rather than melodic choices arises from this goal of pairing solution with resolution. Of course, as is the case with visualization, there is no unique mapping of states of the system to sounds, and so we can consider the results of some different choices of sound assignments and the effects they create.

Ultimately, the goal is to do more than sonify the Tower of Hanoi. The example of the Tower of Hanoi might be thought of as a proof of concept, as I begin to consider the sonification of mathematical puzzles. I close the article with some discussion not only of how one might approach other mathematical puzzles in this way, but also how this approach could turn mathematical puzzles into engines for musical composition.

The Tower of Hanoi, harmonized

The Tower of Hanoi is a mathematical puzzle credited to Lucas [3]. Figure 1 shows an example of a particular version of the puzzle. The premise of the puzzle is that a collection of rings (eight shown in the Figure 1)

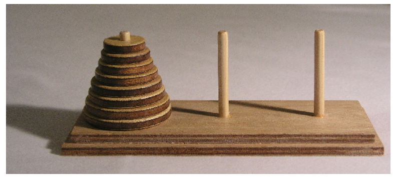

Figure 1: *A Tower of Hanoi puzzle [2]*

must be moved from the pillar on which they sit to one of the other pillars. The rings are to be moved one at a time from one pillar to another, but subject to the condition that a larger ring never be placed above a smaller ring on any given pillar. (There are variations on this basic scheme, but the simplest version of the puzzle will suffice for our purposes.)

To illustrate the application of sonification, we will consider here the case in which the Tower of Hanoi contains only three rings. We will label the pillars from left to right as **X**, **Y**, and **Z**. We refer to the rings as 1, 2, and 3, with 1 the smallest, 3 the largest, and 2 in the middle.

To sonify the state, we assign to each pairing of a ring and a pillar a particular note. For example, ring 2 might be associated with the note B if it is on pillar **X**, $B\flat$ if it is on pillar **Y**, and A if it is on pillar **Z**. Then, at each stage, as the rings are moved, we will have a 3-note chord characterizing the overall state, based on which pillar each ring is on. With the rings starting on the first pillar, initially the notes are **X**1, **X**2, and **X**3. If ring 1 is then moved to pillar **Y**, the chord associated with the next state will be **Y**1, **X**2, and **X**3.

Consider the following solution to the Tower of Hanoi, in which the tower is moved from pillar **X** to pillar **Z**, in a notation based on the above conventions:

$$\begin{aligned}(\mathbf{X}1, \mathbf{X}2, \mathbf{X}3) \rightarrow (\mathbf{Z}1, \mathbf{X}2, \mathbf{X}3) \rightarrow (\mathbf{Z}1, \mathbf{Y}2, \mathbf{X}3) \rightarrow (\mathbf{Y}1, \mathbf{Y}2, \mathbf{X}3) \rightarrow \\ (\mathbf{Y}1, \mathbf{Y}2, \mathbf{Z}3) \rightarrow (\mathbf{X}1, \mathbf{Y}2, \mathbf{Z}3) \rightarrow (\mathbf{X}1, \mathbf{Z}2, \mathbf{Z}3) \rightarrow (\mathbf{Z}1, \mathbf{Z}2, \mathbf{Z}3)\end{aligned} \tag{1}$$

The sonification of this solution then depends on the choice of notes assigned to each of the $\mathbf{X}_i$, $\mathbf{Y}_i$, and $\mathbf{Z}_i$, $i = 1, 2, 3$. As with visualization, some choices lead to more aesthetically pleasing results than others, and some likewise provide a clearer sense of the trajectory of the solution. Out of various choices, here are three options that stood out, where we denote the set $\{\mathbf{X}1, \mathbf{X}2, \mathbf{X}3\}$ as $\vec{\mathbf{X}}$, and analogously for the other pillars:

(i) Chromatic descent: $\vec{\mathbf{X}} = (A, F\sharp, D)$ $[D \text{ maj}]$, $\vec{\mathbf{Y}} = (G\sharp, E\sharp, C\sharp)$ $[C\sharp \text{ maj}]$, $\vec{\mathbf{Z}} = (G, E, C)$ $[C \text{ maj}]$

(ii) V7 to I via diminished chord: $\vec{\mathbf{X}} = (F, D, B)$ $[G7]$, $\vec{\mathbf{Y}} = (E\flat, A, F\sharp)$ $[F\sharp \text{ dim}]$, $\vec{\mathbf{Z}} = (G, E, C)$ $[C \text{ maj}]$

(iii) V to I via IV7: $\vec{\mathbf{X}} = (A, F\sharp, D)$ $[D \text{ maj}]$, $\vec{\mathbf{Y}} = (B\flat, E, C)$ $[C7]$, $\vec{\mathbf{Z}} = (D, B, G)$ $[G \text{ maj}]$

Each of these will, to a listener familiar with the major and minor keys of the *tonalité moderne* [8], provide a sense of a journey from a starting point to a suitable resolution with tension along the way as the puzzle gets solved. The latter provide a resolution of a major fifth to the tonic, while the first associates movement from one pillar to the next with chromatic half-tone steps downward.

When applied to the solution of the puzzle given by Sequence 1, these various note assignments produce the chord sequences displayed in Figures 2, 3, and 4.

Figure 2: *Chord sequence (i)*

Figure 3: *Chord sequence (ii)*

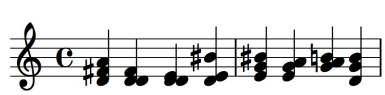

Figure 4: *Chord sequence (iii)*

Three observations are in order here. First, these distinct note assignments produce very different sonic experiences even though they arise from the same solution to the same puzzle, with the choice of note assignments impacting both the aesthetic appeal and informational clarity of the sonification. (There were other choices I tried that seemed harmonically sensible but were underwhelming in terms of their output.) Second, even within this particular sonification framework, there are other degrees of freedom that could be used to communicate additional information. For example, one could attach a different instrument's timbre to each pillar, so that one hears not only a change in note as, say, ring 1 moves from pillar **X** to pillar **Y**, but a change in instrument as well.

Third is that we can view this not simply as a method to provide an auditory representation of the solution to the puzzle, but the relationship can be inverted: one could use manipulations of a mathematical puzzle as a compositional tool. This would be relatively straightforward to implement within the context of computer renditions of this or other similar puzzles.

Other puzzles, other sound choices

Of course, sonification of mathematical puzzles is not limited to the Tower of Hanoi. Here, I discuss two other puzzles. In presenting these examples, the idea is to examine some of the pros, cons, and limitations of sonification choices, not to suggest that the choices presented are unique or exhaustive.

Consider the problem of constructing a knight's tour on a chessboard [6]. A straightforward sonification would be to assign a distinct note to each square. One would begin with all 64 notes playing at once, and then, as each square is visited, its associated sound would stop playing. Success occurs when the entire board has been visited and, simultaneously, silence is achieved. While it would be clear sonically when one has completed a knight's tour, and indeed even when there are just a few squares left, the sonic experience does not seem musically interesting, and so while this provides (at least toward the end of stages of a tour) a reasonable measure of informational clarity, it is a challenge to make this aesthetically satisfactory. One way to obtain a more meaningful sonic structure would be to select an orchestral piece with 64 parts, assigning one part to each square, and then use either a subtractive approach as above, or an additive approach, building up the whole piece from nothing during a complete tour.

A very different example is provided by the puzzle known as Untangle [9]. In this puzzle, which is only effectively implemented on a computer, one has a set of vertices in a two-dimensional region, with some pairs connected by an edge. In the initial configuration, some number of edges will cross each other. One is then tasked with moving vertices, one at a time, with the goal of reaching a configuration in which none of the connecting lines cross each other.

An appealing approach to sonification for this puzzle is based on the number of line crossings in each configuration. Let the initial number of line crossings be $N(0)$, and the number of line crossings after j moves be $N(j)$. One assigns a chord K_N to each possible value of N. To create a sense of musical resolution with the actual solution, one might treat K_0 as the tonic, with other chords being assigned by their relation to the tonic. For example, one possibility is that chords associated with a small number of line crossings might be V7, IV, or vi, while one could choose chords harmonically further afield for larger numbers of line crossings, so that as one gets closer to a solution, one gets closer to a musical resolution.

For an example, consider Figure 5, in which we see a series of images representing different stages of a solution of a 6-vertex Untangle puzzle. With the sequence of vertex crossing numbers of $\{3, 3, 4, 2, 0\}$, we can get a series of chords, depending on the chord assignments specified. The interesting piece is not what to do for this one example, but rather to make chord assignments that can be applied across distinct instances of the puzzle, producing a variety of musical experiences.

For instances with more vertices, we would need more chords in our repertoire of possibilities. This might lead to a more chromatic final result, and it would be worthwhile to maintain strong harmonic relationships among chords associated with values of N of around the same magnitude. The variety and sequence of chords one encounters conveys something about the size of the problem, and whether the number of line crossings

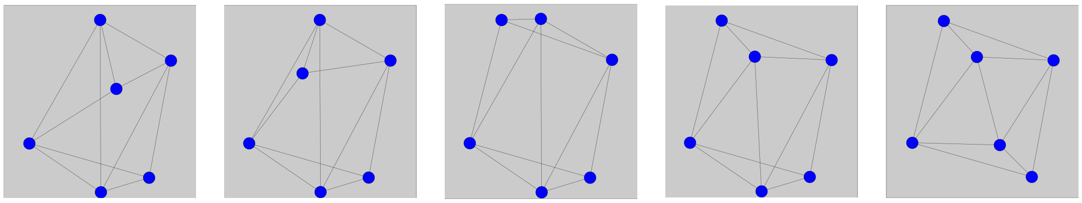

Figure 5: *A sequence of Untangle positions.*

decreases monotonically to zero or involves some increase and decrease along the way.

But what is most appealing to me about this is that for this puzzle, it is very easy to imagine the role between the puzzle and music being reversed. Rather than just thinking about solving the puzzle, this sonification could turn the puzzle into an instrument. Allowing the chords to be generated in real time, one could simply play with moving the vertices not to get a zero-crossing configuration, but simply to generate pleasing sounds. I look forward to the realization of such a novel musical instrument.

Conclusion

In this paper, I constructed harmonically grounded sonifications attached to the Tower of Hanoi puzzle, and offered some thoughts on the sonification of other mathematical puzzles. Clearly, this paper is not meant to be exhaustive, but rather to suggest ways we can forge meaningful connections between mathematical puzzles and music. The idea that puzzles could be turned into compositional tools, and even into dynamical instruments, takes the idea of sonification into a realm beyond that of providing informational clarity and into one of creating aesthetic objects that are valuable in their own right.

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