Euler’s Polyhedron Formula for Uniform Tessellations

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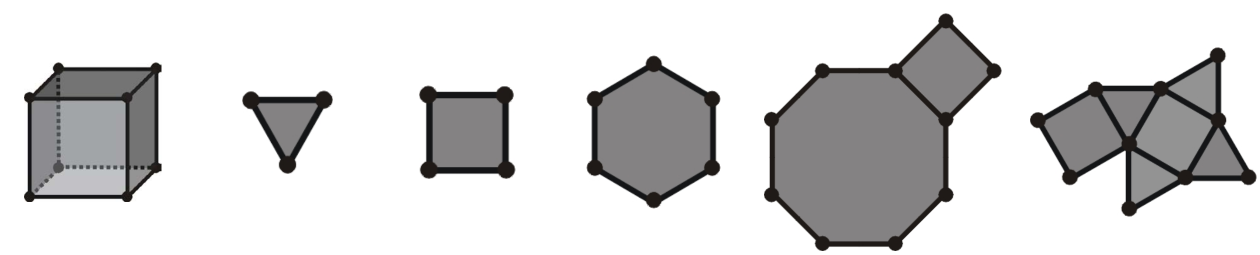
Abstract

In 1758 Euler stated his formula for a polyhedron on a sphere with V vertices, F faces and E edges: $V + F - E = 2$. Since then, it has been extended to many more cases, for instance, to connected plane graphs by including the ‘exterior face’ in the number of faces, or to ‘infinite polyhedra’, using their genus g. For $g = 1$ the latter generalization becomes $V + F - E = 0$ or $V + F = E$ and that is the case for uniform planar tessellations. This way, two popular topics, uniform planar tessellations and Euler’s formula, can be combined.

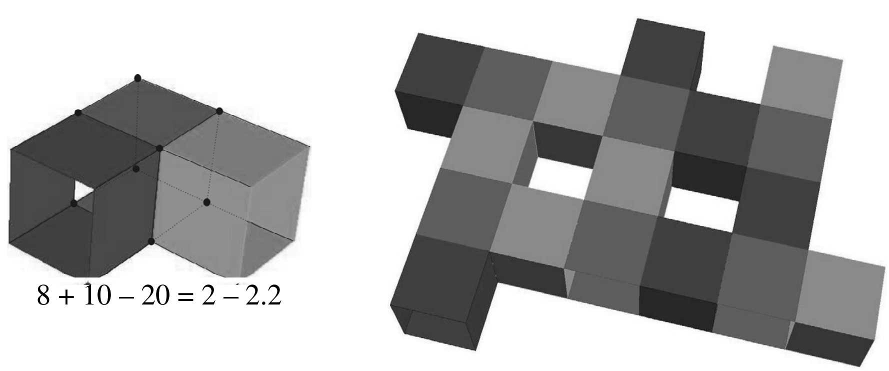
Euler’s Polyhedron Formula

Euler’s formula for a polyhedron on a sphere with V vertices, F faces and E edges states that $V + F - E = 2$. For instance, for a cube, $V = 8$, $F = 6$ and $E = 12$ and indeed $8 + 6 - 12 = 2$. It can be generalized to convex polyhedra. In the plane too, it can be applied to any connected graph. Its edges (arcs) partition the plane into a number of faces, one of which is unbounded and called the ‘exterior face’. For instance, for a triangle, $V = 3$, $F = 2$ (the triangle plus the exterior face) and $E = 3$ so that $3 + 2 - 3 = 2$. Some other cases, such as the square and the hexagon, or the compound figures of an octagon together with a square and of two squares and three triangles are shown in Figure 1 (the examples were chosen because they will be applied later).



$8 + 6 - 12 = 2$ (a) $3 + 2 - 3 = 2$ (b) $4 + 2 - 4 = 2$ (c) $6 + 2 - 6 = 2$ (d) $10 + 3 - 11 = 2$ (e) $10 + 7 - 15 = 2$ (f)

Figure 1: *Some classical straightforward examples of Euler’s formula.*

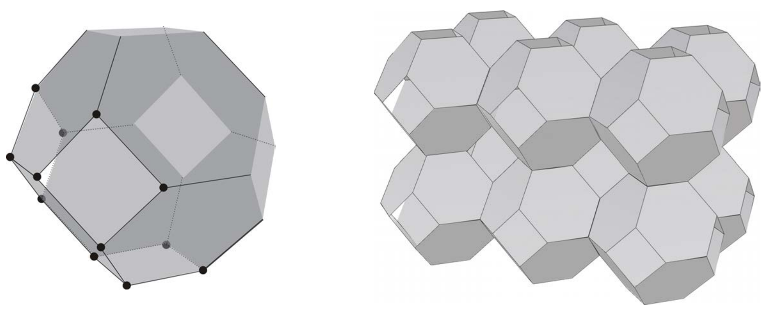


(a) (b)

Figure 2: *An infinite polyhedron of genus 2.*

This can be found in many textbooks (see for instance [1], [2]). For higher genus infinite 'polyhedra' (see [9]), a generalization states that $V + F - E = 2 - 2g$, where g is the genus. If the latter is defined as the number of 'holes' an infinite polyhedron's genus would be infinite, but the formula applies to a repeat or translation unit (see [3]). For exact definitions and even further generalizations – beyond the scope of this present paper –, we refer to [5].

For instance, if parts of three cubes are connected to form an element of an infinite structure, they can be put together so that they form a layer of genus 2 (see Figure 2). We only count the 8 vertices of the middle cube, as the other cubes are connected to other elements. The 6 horizontal faces plus 2 times 2 vertical faces of the outer cubes give 10 faces all together. The rest of the cubes remains open, as the elements are connected. We count all 12 edges of the middle cube, plus 2 times 4 edges of the outer cubes standing perpendicularly on that middle cube, but not those where they will be connected to the other elements. Thus, $V + F - E = 8 + 10 - 20 = -2 = 2 - 2.2 = 2 - 2g$.



(a) (b)

12 + 8 – 24 = 2 – 2.3

Figure 3: *An infinite polyhedron of genus 3.*

Also, a truncated octahedron without its squares can be put together so that the remaining hexagonal faces form an infinite polyhedron of genus 3 (see Figure 3). We count the 3 times 4 vertices of 3 squares (the one to the left, on the bottom and in the front), but not those of the other squares as the elements will be connected. There are 8 hexagonal faces. Each of the three squares on the left, the bottom and the front has 4 edges, and there are 4 edges pointing towards the top square, 4 to the bottom square, and 4 horizontal ones, or 24 edges in total. Thus, $V + F - E = 12 + 8 - 24 = -4 = 2 - 2.3 = 2 - 2g$. Other examples can be found in [6] and [7], but the situation for uniform tessellations is more straightforward as the flat constructions in the plane can be more easily represented.

The Formula

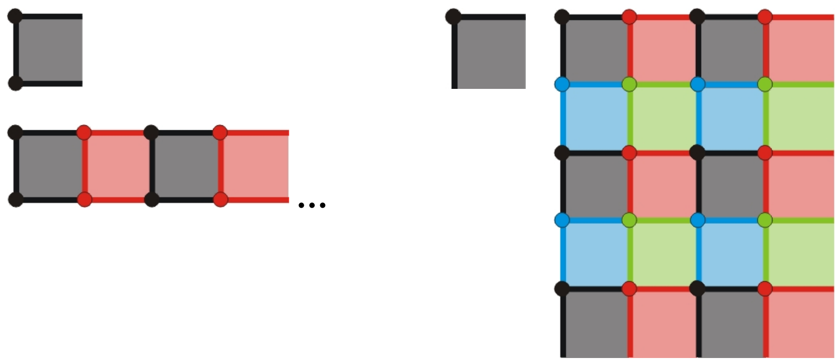
A uniform tessellation in the plane can be seen as a $g = 1$ case of the previous formula, $V + F - E = 2 - 2g$, and thus $V + F - E = 0$. We can also understand the formula intuitively by a two – step procedure. First, we note that we can reduce the case to a single polygon (as in Figures 1(b), (c) and (d)), since compound polygons (as in Figures 1(e) and (f)) reduce to a single polygon by removing edges, but then the number of faces reduces as well and the sum $V + F - E$ remains invariant. Next, we can make a linear chain by repeating that single polygon. The removal of one edge corresponds to the reduction of the number of vertices by 2. Thus, the initial formula $V + F - E = 2$ for the original polygon becomes $V + F - E = 1$ for the adapted polygon, the element of the chain of polygons. Going in one more direction will at some point take only one vertex away, as one vertex will be common to both directions (the 'corner'), and so $V + F - E$ will not change during this operation. However, the number of faces F diminishes by 1, because the outer face doesn't have to be added anymore, as a uniform planar tessellation covers the entire plane. Thus, $V + F - E = 0$.

Examples

The colors are merely indicative, to make the counting of the vertices and edges easier.

Example 1: Tessellation of Squares

Following the above intuitive two–step procedure, we first build a series of adjacent squares extending indefinitely to the right: 2 vertices and 1 edge are removed and thus the formula given in figure 1(c) becomes $V + F - E = 2 + 2 - 3 = 1$. To build a two–dimensional tessellation, only 1 vertex and 2 edges are needed, because of the additional adjacent squares. When the entire plane is filled, the outer face should not be added anymore. Thus, in this case: $V + F - E = 1 + 1 - 2 = 0$.



(a) (b)

Figure 4: *Making a chain of squares (a) and a square tessellation (b).*

Example 2: Hexagonal Tessellation

For a series of adjacent hexagons, only 4 vertices and 5 edges need to be counted: $V + F - E = 4 + 2 - 5 = 1$, and if an entire tessellation fills the plane, only 2 vertices and 3 edges: $V + F - E = 2 + 1 - 3 = 0$.

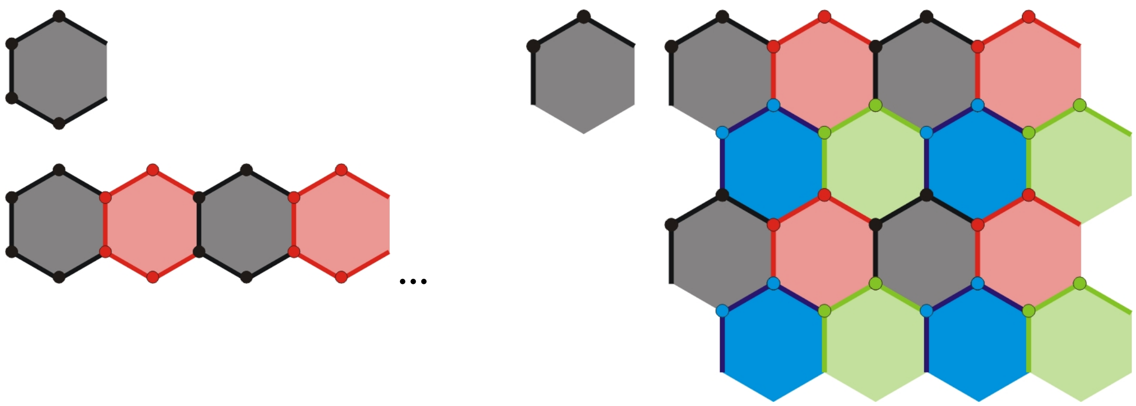


Figure 5: *A chain of hexagons (a) and a hexagonal tessellation (b).*

Example 3: Triangular Tessellation

For a linear arrangement, 1 vertex and 2 edges suffice: $V + F - E = 1 + 2 - 2 = 1$. In two dimensions we expect to remove more vertices, as in the previous examples. That would make it awkward as there would be no more vertices left. So, we combine two triangles and keep 1 vertex and 3 edges: $V + F - E = 1 + 2 - 3 = 0$.

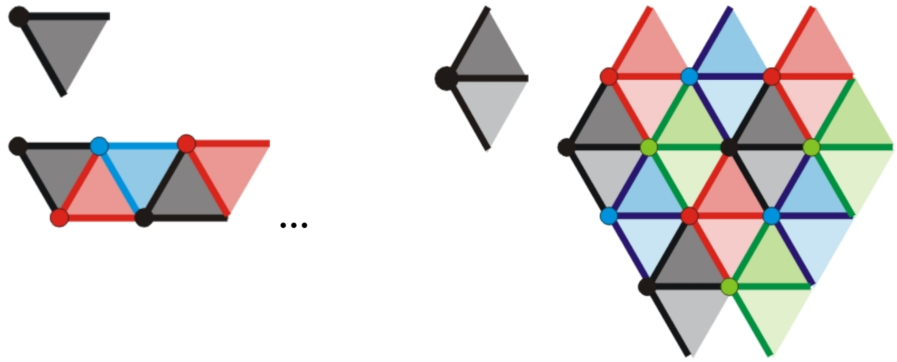


Figure 6: *A chain of triangles (a) and a triangular tessellation (b).*

Example 4: Truncated Square Tiling
For the linear arrangement, $V + F - E = 7 + 3 - 9 = 1$, while for the plane filling tessellation: $V + F - E = 4 + 2 - 6 = 0$.

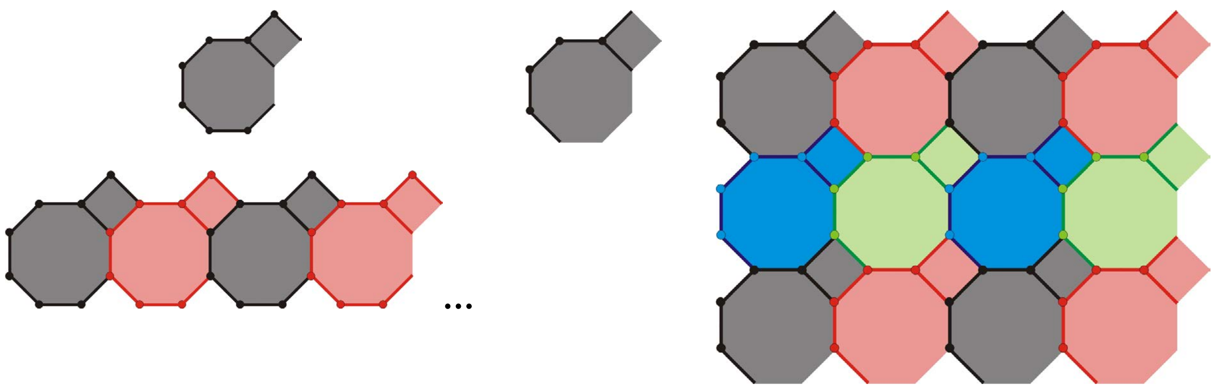


Figure 7: *A chain of truncated squares and squares (a) and a truncated square tiling tessellation (b).*

Example 5: Snub Square Tiling
Due to the limited space, we immediately consider the more interesting entire tiling, made by repeating 2 squares and 4 triangles, minus 6 vertices and 5 edges: $V + F - E = 4 + 6 - 10 = 0$.

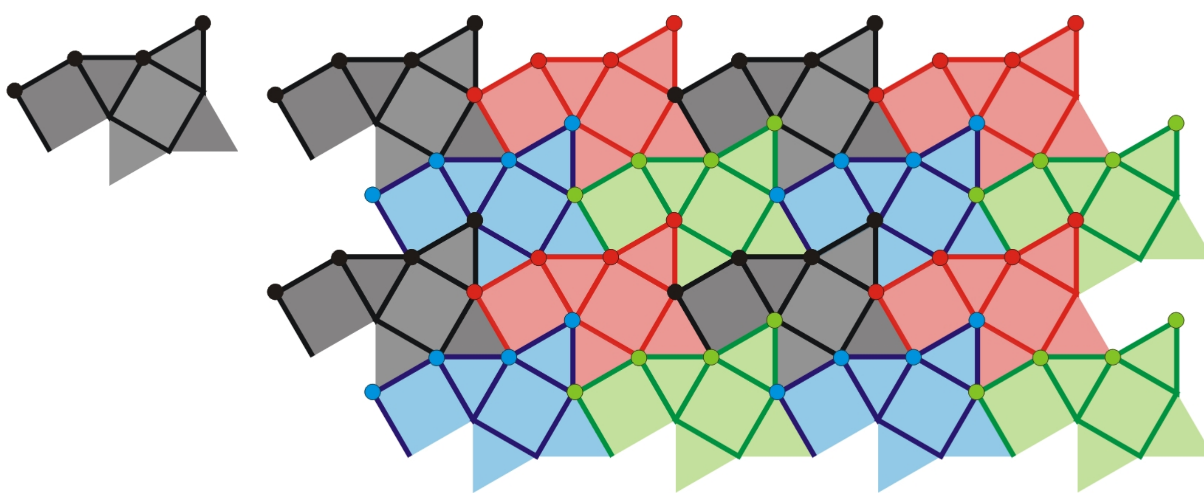


Figure 8: *The snub square tiling.*

Example 6: Truncated Trihexagonal Tiling

It looks as if one element of the arrangement would have 14 faces (6 squares, 6 hexagons, 1 dodecagon and 1 exterior face) but because of the surrounding elements, 3 squares and 4 hexagons can be omitted (as well as the exterior face, as was done for all of the above tessellations): $V + F - E = 12 + 6 - 18 = 4 + 6 - 10 = 0$.

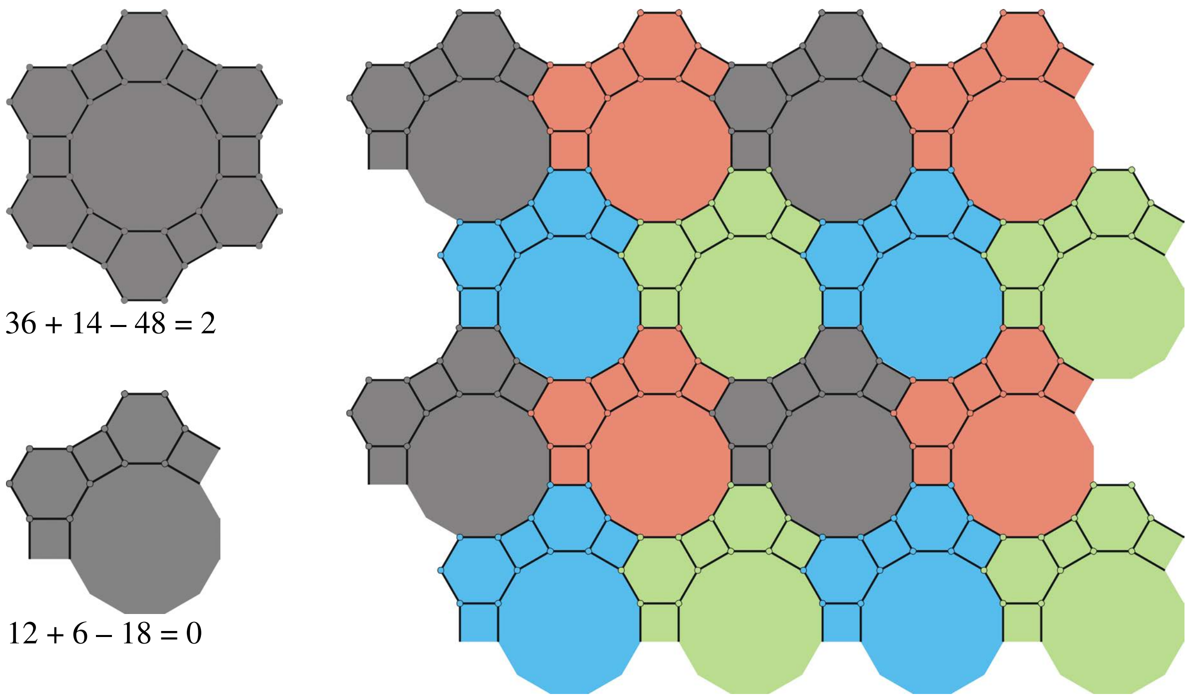


Figure 9: *Truncated trihexagonal tiling.*

Summary and Conclusions

Tessellations are a fun topic to teach and occur in many college handbooks, even in textbooks addressed to liberal arts math students (see for instance [8]). They also are an inspiring topic for artwork. Here is an example taken from the cover of [4]: consider the irregular pentagons in the given arrangement. When grouped as suggested, the counting is similar to the situation above, though the example does look different: $V = 13$, $F = 1$, $E = 14$, and $13 + 1 - 14 = 0$ indeed.

Now Euler's formula is quite surprising too, especially in the case of planar tessellations: $V + F = E$, or "the sum of the number of vertices and faces equals the number of edges". Such an easy property can be taught at any level. One doesn't even need a 'formula with symbols' as the property can be easily expressed in words. Thus, it was thought it could be of interest to emphasize this remarkable yet often overlooked property of uniform tessellations. It can be a steppingstone for further investigations, noting, for instance, that the formula is the same for tessellations on a donut.

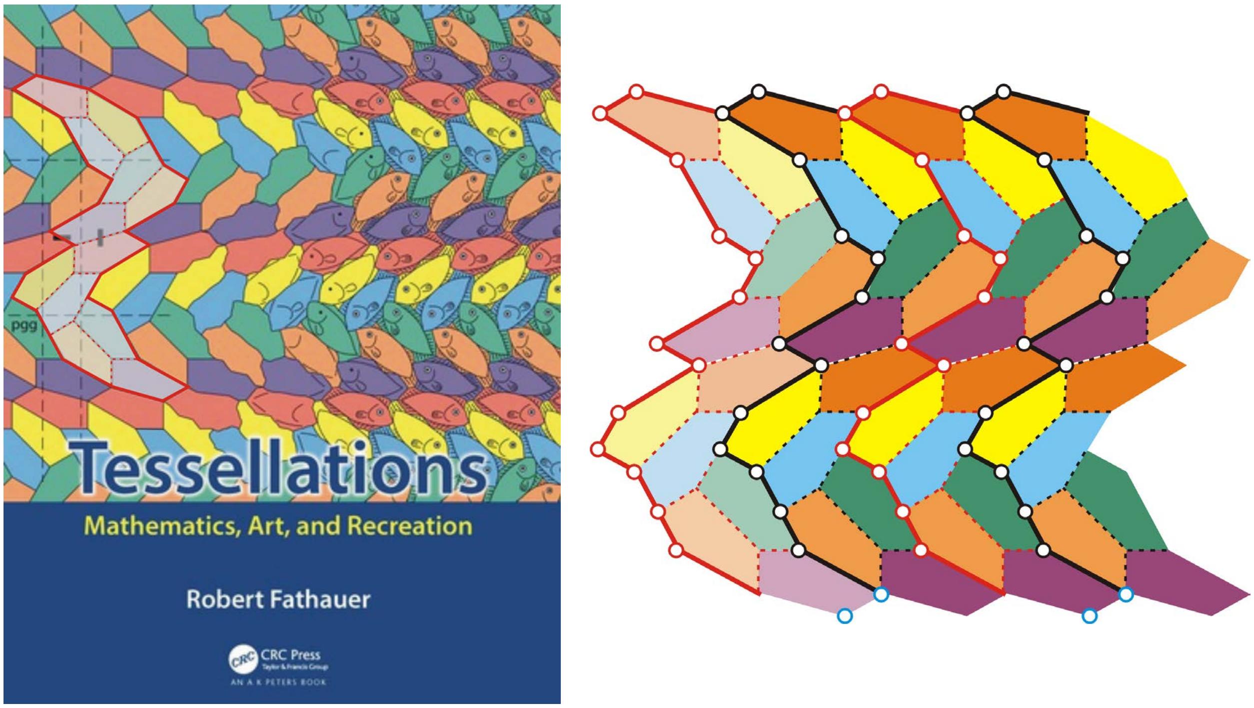


Figure 10: *The formula can also be applied to fun examples.*

References

[1] J.H. Conway, H. Burgiel, Ch. Goodman – Strauss, *The Symmetries of Things*, A K Peters Ltd., 2008.

[2] P.R. Cromwell, *Polyhedra*, Cambridge University Press, 1999.

[3] S. Dutch, *Hyperbolic Tessellations*, http://superliminal.com/geometry/infinite/infinite.htm, consulted on 11 May 2020.

[4] R. Fathauer, *Tessellations: Mathematics, Art, and Recreation*, A K Peters/CRC Press; 1st edition, December 8, 2020.

[5] B. Grunbaum, G. C. Shephard, "A New Look at Euler's Theorem for Polyhedra", *The American Mathematical Monthly*, Vol. 101, No. 2, Feb. 1994, pp. 109 – 128

[6] D. Huylebrouck, "An Euler – Cayley Formula for General Kepler – Poinsot Polyhedra", *Geometrias'19: Book of Abstracts*, Vera Viana (Ed.), Porto, Aproged, 2019, pp. 99 – 104.

[7] D. Huylebrouck, "A New Regular (Compound) Polyhedron (of Infinite Kepler – Poinsot Type)", *The American Mathematical Monthly*, Vol. 124, n° 3, March 2017, pp. 265 – 268.

[8] D. Sobecki, *Math in Our World*, McGraw – Hill Education, 4th edition, 2018, 944 pp.

[9] A. Wachman, M. Burt, M. Kleinman, *Infinite Polyhedra*, Technion Publication, Faculty of Architecture and Town Planning, 1974, 2005, 102 pp.