Sculptable Kaleidocycles: Visualizing Variable Cell Geometry

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Abstract

Kaleidocycles, a class of ring linkages, have long been depicted as compositions of ordinary tetrahedra. However, this paradigm only depicts one part of the picture; it has been adopted only for the mathematical and graphical facility it provides. In reality, there are infinitely many geometries which satisfy the parameters of a kaleidocyclic base unit (a “cell"). The purpose of this paper is to solidify that concept by improving visualization and fabrication techniques for these "sculpted" kaleidocycles. To achieve this, ordinary kaleidocycles were first simulated in a mesh-driven environment, where their base cells were then sculpted through the use of set operations (intersection and subtraction, in particular) with other solids. The results are intriguing new objects with artistic and mechanical implications. These articulations upon ordinary kaleidocycles allow further specialization and customizability of the mechanisms, improving their applicability to various tasks.

Introduction

Figure 1: *a kaleidocycle at $t = 0, \pi/2, \pi,$ and $3\pi/2$ radians of toroidal rotation*

Since the 1950s, artists, mathematicians, and children alike have enjoyed the wonders of kaleidocycles. Kaleidocycles are ring-like linkages composed of smaller modules–“cells"–which are hinged together at opposite edges. The kaleidocycles we treat have the interesting property that they can undergo continuous, toroidal rotation (Figure 1).

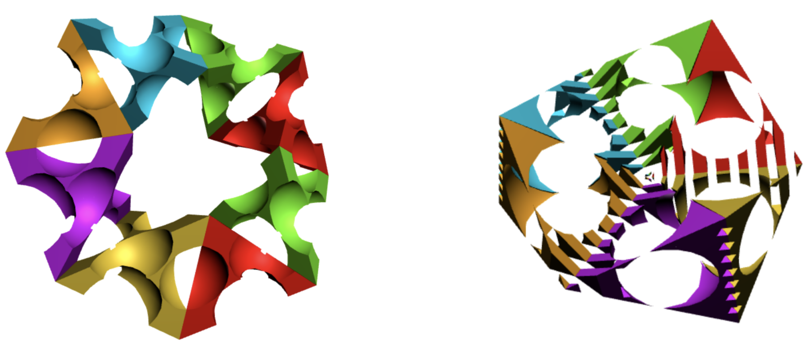

Figure 2: *some intricately sculpted kaleidocycles*

However, kaleidocyclic structures composed of more interesting, non-tetrahedral cells have been prohibitively difficult to visualize and construct. We have developed a more effective and dynamic method to simulate such kaleidocycles. Here, we apply constructive solid geometry (CSG), a solid modeling technique for polyhedral set operations, to deform and “sculpt" basic cells, generating new geometries to serve as the bases for kaleidocycles (Figure 2).

Methods in Kaleidocycle Simulation

In order to make set operations possible on the base units of kaleidocycles, they first had to be simulated as mesh objects. That is, in the computer, the kaleidocycles had to be represented as compositions of triangles due to the requirements of the CSG algorithm employed ([4], discussed by Laidlaw [3] and Segura [6]). To do this, we extended and modified the methods detailed by Engel [1]. First, a "principal" cell was rotated. Then, the remaining cells were rotated based on their relationship to the principal cell.

Rotating a Single Cell

The cell in the first octant with edge on the x-axis is designated as the "main" or "principal" cell and is treated first (Figure 3, left). For this, the orthonormal vectors $\vec{u}, \vec{v}$, and $\vec{w}$ (Figure 3, right) are defined to correspond to the motion of segments $\overline{AB}$, $\overline{CD}$, and $\overline{PQ}$, respectively. They are then used in a change of basis matrix applied to the vertices of the cell. Vectors $\vec{u}, \vec{v}$, and $\vec{w}$ are parameterized as in [1]. Thus, for each vertex of the tetrahedron

$$T(\vec{x}, t) = \begin{bmatrix} | & | & | \\ \vec{u}(t) & \vec{w}(t) & \vec{v}(t) \\ | & | & | \end{bmatrix} \vec{x} + \vec{M}(t).$$

This represents a change of basis and a translation to the point M, the parameterized centroid of the cell. Initially, as in Figure 3, $\vec{u} = \vec{e}_1, \vec{w} = \vec{e}_2$, and $\vec{v} = \vec{e}_3$, meaning no rotation has occurred. Therefore, $T(\vec{x}, 0) = \vec{x} + \vec{M}$. In all, T describes the orientation and position of the principal cell as a continuous function of toroidal rotation angle t. However, to move between discrete points in rotation, such as frames in an animation, $T(t_{n-1})$ must be inverted before $T(t_n)$ can be applied.

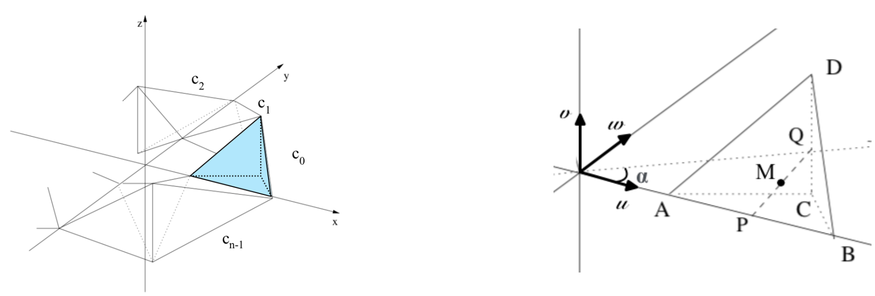

Figure 3: *left: the principal cell, c_0, highlighted in context; right: definitions of $\alpha, \vec{u}, \vec{v}, \vec{w}$; both adapted from [1]*

N-Specific Transformations

Once the movement of the principal cell is described, rendering the full kaleidocycle follows rather naturally. To do so, we introduce the concept of n-specific transformations: transformations based on the index of a cell in the kaleidocycle. We define a cell's index by beginning at 0 with the principal cell and counting up in an anticlockwise direction. As such, the cells adjacent to the principal cell c_0 in an n-cell kaleidocycle are always c_1 and c_{n-1}.

Following this numbering, it becomes clear that for even values of i, c_i can be obtained by revolving c_0 $i\alpha$ radians around the z-axis (α as in Figure 3, right). For odd values of i, c_i can be obtained by reflecting c_0 over $y = xtan(\alpha)$ and rotating by $(1-i)\alpha$ around the z-axis [1]. Thus, each cell can be very easily represented as a reflection of the principal cell, rotation of the principal cell, or a combination of the two.

Application of CSG to Cells

With the cells simulated in mesh form, it becomes possible to apply the polyhedral set operations intersection and subtraction. This allows them to be combined with other solids, to "sculpt" the blank, standard tetrahedra as in Figure 4. Our software includes four tools – a cube, a sphere, a cone, and a cylinder – each of which can be scaled on 3 axes. However, many more tools can be used in this same process.

For physical manufacturing, each hinge must remain, at the least, as a single line segment. Should this requirement be violated in sculpting, with hinges reduced to singular points, they will effectively become ball joints. Additional degrees of freedom will be opened in the resulting ring linkage, straying from the definition of a kaleidocycle.

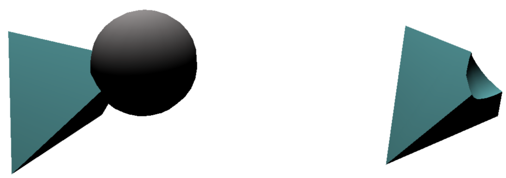

Figure 4: *left: a spherical tool overlapping a cell's vertex; right: a new cell created after a subtraction operation*

Results

Following are the results of the process. As previously described, the cell is first constructed through a series of CSG operations, and then a new, sculpted kaleidocycle is constructed. Some of these "sculpted" ring linkages have interesting properties and implications which are expanded on in the next section.

1. **Spherical Cuts**

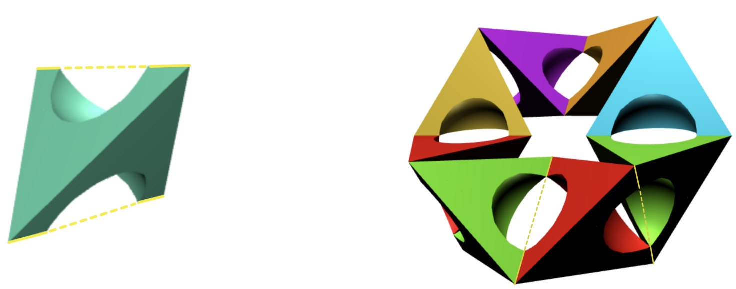

Figure 5: *left: a cell from an $n = 8$ kaleidocycle after two spherical sectors are removed. The remaining hinge portions are marked by solid lines while the removed portions are marked by broken lines; right: an $n = 8$ kaleidocycle built from the cell.*

Spherical cuts, as in Figure 5 (right), give the interesting effects of "holes" and "stoppers." On the top face of the kaleidocycle, we see four holes formed by the joined spherical sectors. In contrast, along the outer edge of the structure (see bottom-right), the sectors come together in a scoop-like shape as a "stopper," or a closed hole. As this kaleidocycle undergoes toroidal motion, these "holes" and "stoppers" alternate position, opening up possibilities for this structure as a filter or valve.

2. **Prismatic Cuts**

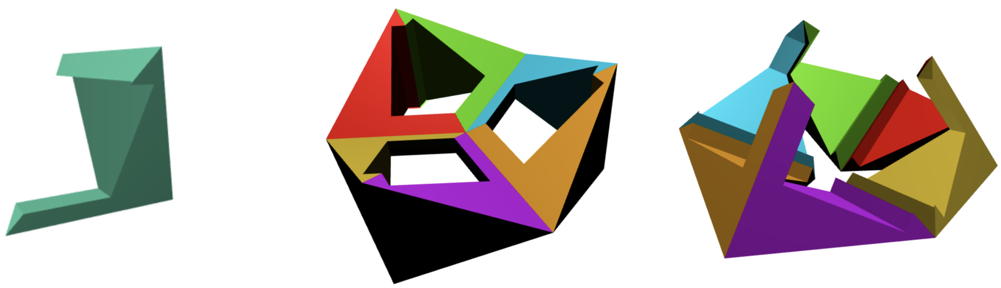

Figure 6: *left: a cell from an* $n = 6$ *kaleidocycle after a rectangular prism is subtracted. Both hinges (top and bottom edges) remain fully intact; center: an* $n = 6$ *kaleidocycle built from the cell; right: the kaleidocycle at another point in its rotation*

The “legs" protruding from the cell in Figure 6 (left) can be seen meeting at the center of the resultant kaleidocycle (center). As expected, these legs separate as the linkage rotates into different orientations (right).

3. **Conical Cuts**

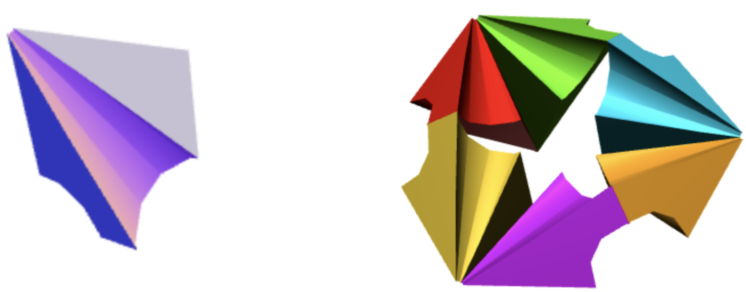

Figure 7: *Left: a cell from a* $n = 6$ *kaleidocycle after undergoing two conical cuts along its edges; right: an* $n = 6$ *kaleidocycle built from the cell.*

The conical cut shown in Figure 7 is a versatile tool for artists and engineers alike, allowing a varying width and depth to be achieved with a single stroke. In this example, the "conical" cut is truly a pyramidal cut; three triangular faces are visible in the cut. To optimize the toroidal animation, our program represents curved surfaces at a lower resolution. The resolution can be increased in production.

Discussion

The method we propose is based on existing methods in both solid modelling and kaleidocycle simulation. In bringing the concept of constructive solid geometry, a common technique in CAD, to the world of kaleidocycles, a streamlined and intuitive method of visualizing various cell geometries is created.

By following this method, a user can not only construct novel kaleidocycles digitally, but also view their toroidal motion almost instantaneously without changing softwares (see our JavaScript-based online tool, [8]). This removes many of the barriers that previously stood in the way of such visualizations. With this increased access, kaleidocycles take on various new meanings and use cases.

It is worth noting that curved cuts using CSG can be considerably more computationally expensive. In tetrahedron-based kaleidocycles, there are always 4 vertices per cell. Regardless of how many cells the structure contains, this restriction on the number of vertices keeps the total number of matrix operations low. In contrast, the cell of a sculpted kaleidocycle can have an unlimited number of vertices. Of course, on a computational level, a smooth cut is modeled by a large number of points in close proximity, greatly increasing the number of matrix operations required in simulation. To combat this, a middle ground between perfectly curved appearance and performance is often preferred for simulation purposes. For example, curved sculpting tools used in the CSG operations such as spheres, cones, and cylinders are often created at a slighly lower resolution such that their curved appearance is maintained, but the computational load of simulation is reduced. Should the kaleidocycle be fabricated through the use of a 3D printer or otherwise, the resolution of these curved surfaces can be again increased.

Mechanically and artistically, sculpted kaleidocycles are more intriguing than standard, tetrahedron-based models. Figure 6 (a kaleidocycle formed by a single prismatic cut) describes a mechanism with interesting implications. By interpolating between the two states shown (center and right), the six-hinged kaleidocycle functions very similarly to a common mechanical claw, often seen in arcade machines and robotics applications. In the center image, the mechanism is in the "closed" state, while on the right, it is "open", meaning that a smooth motion between these states would be a "grasp." When the same sculpting methods we detail here are applied to other types of kaleidocycles, namely Möbius Kaleidocycles [5] (kaleidocycles with "one side"), many of the applications detailed by Schönke et al. such as self-propelling aquatic devices, mixing mechanisms, and others can be further extended by the additional control and customizability over the structure provided. Kaleidocycles' artistic potential is also unbounded, allowing a new creativity to be brought to what has been, at best, a tedious process in the past.

Future Work

The program currently supports STL exports for 3D printing. The following considerations apply to the fabrication of these structures using 3D printing technology or any other method.

Volume and Material Considerations

The question of the volume of material required to produce a physical kaleidocycle model may arise, whether for production constraints or otherwise. To compute this value for an ordinary kaleidocycle is straightforward given the simplicity of the component cells (it is simply a multiple of the volume of a tetrahedron). However, for the more complex, sculpted kaleidocycles, this can be more involved. In this case, we propose Zhang and Chen's method of volume calculation for mesh objects [7]. It defines mesh volume as the sum of the *signed volumes* of the tetrahedra formed by each surface triangle and the origin, where the sign is determined by the location of the origin with respect to the surface normal. That volume can be multiplied by n to obtain the total volume of the model.

Production of Hinges

For visual simplicity and clarity, hinge mechanisms have been omitted from the simulation. However, should models from this tool be directly exported to CAD software as previously suggested, hinges would be required for mechanical functionality. These could be added prior to printing by the user or added onto the fabricated products themselves. In our case, we used flexible adhesive tape to hinge the structures. Alternatively, if a simple male-female hinge design is employed, one of each type could be attached to the ends of a sculpted base unit, allowing several to be hinged together to create the physical kaleidocycle.

Texture Mapping

While Schattschneider and Walker's 1977 book, *M.C. Escher Kaleidocycles*, articulated the surface designs of kaleidocycles, our work articulates their form. However, there is no reason why a computer simulation could not address both topics. To do this, our program would need to be expanded to allow texture mapping, the application of various textures to the surface of a mesh object. With that functionality, an artist or sculptor could hope to expand even further the possibilities for these structures.

Returning to Paper Models

There is a small subset of sculpted kaleidocycles that can still be viably constructed from paper. For example, the kaleidocycle in Figure 6. While difficult, this kaleidocycle is comprised of straight lines and folds, and thus a viable net could be created for it. To accommodate paper as a medium, net unwrapping functionality could be added to the software. Several algorithms exist for this.

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