String Mechanism for Polyhedral Pop-up Card Design

Lauren H. Li

Massachusetts Institute of Technology, Cambridge, USA; lhli@mit.edu

Abstract

We present a novel string mechanism for the design of pop-up cards. We apply this mechanism to create three new pop-up card designs: (1) an asymmetrically folding cube, (2) a rhombicuboctahedron, and (3) a sliceform character, all of which use string to induce the pop-up. Easily adaptable, our technique can be used in future pop-up card design. Moreover, the questions we pose in the process may have even broader implications in robotics and engineering.

Introduction

Following a renaissance of pop-ups books in the mid to late 20th century, the designs of pop-up books and pop-up cards have soared to new heights, with an ever-growing collection of design gadgets and techniques [1]. Here, we present a pulley-like string mechanism as a new technique for pop-up card design. Previously, pulleys have been used to raise pop-up structures of plastic bricks [9], and existing pop-up cards make use of strings to fix edges of sliceforms [12], turn an axle [2], and form a hexahedron at 360° [3]. However, we find our use of string and pulley mechanism an entirely new way to construct 180° pop-up cards. We show our mechanism in two polyhedral card designs: a cube that collapses with respect to a non-planar symmetry and an Archimedean solid. In the process, we pose an open problem on the existence of symmetric nets for polyhedra, which relates to the problem of polyhedron unfolding [5, 6]. We also show that our string mechanism can be applied to sliceform-based cards. To the best of our knowledge, the mechanisms and designs presented in this paper are novel, and our results may aid in the design of future pop-up cards.

String Mechanism

The key feature in each of our pop-up designs is a pulley-like mechanism, which we call the *string mechanism*. This mechanism functions such that when the card is open, the string is taut, and when the card is closed, the string is loose. That is, we use a string of length $2\ell + x$, where ℓ is the width of each cover of the card and x is the distance from some point p on one cover to a point q on the edge of that same cover (Figure 1). The point q functions as the fulcrum of the pulley; the string is attached to the pop-up structure, threaded through the point p, and attached to the card at a point q', which is the point q reflected over the crease of the card. As such, when the card is closed, there exists a length 2ℓ segment of loose string. This guarantees that the pop-up structure attached at the point p gets pulled to its desired position when the card is opened but can also move freely when the card is collapsed. We show this mechanism is effective for a variety of designs.

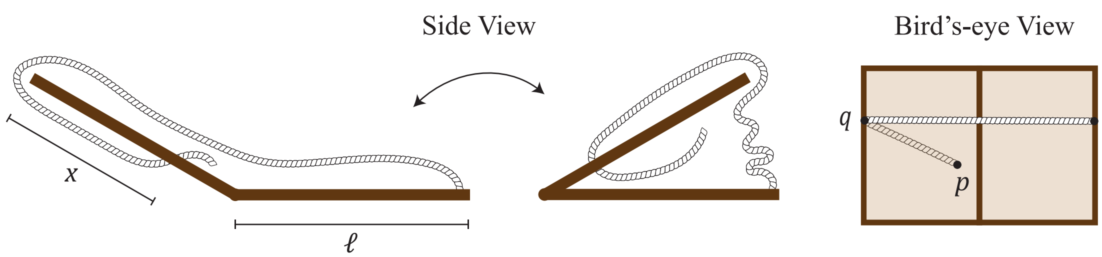


Figure 1: *String mechanism applied in our designs.*

An Asymmetrically Folding Cube

We use the string mechanism to create a pop-up card of an asymmetrically folding cube. This design is inspired by Peter Dahmen's nested cubes pop-up card, where the cubes fold across the diagonal planar symmetry [4]; we have not seen any polyhedral pop-up cards that fold with respect to non-planar symmetries.

We first define the *fold plane* as the plane that slices through the crease line and is perpendicular to the surface of the card opened to 180°. In this language, we seek to design a pop-up card of a cube such that the fold plane of the card is *not* a planar symmetry of the cube (Figure 2). Observe that for a plane cutting asymmetrically through the cube, the horizontal cross section is not bilaterally symmetric across the *fold line*, the line projection of the fold plane. Hence, we cannot glue more than one edge of the cube's square base to the card; moreover, we cannot fix one of the edges that intersect the fold line. We can, however, fix an additional point on the opposing side of the fold line and still avoid collisions of the paper with itself.

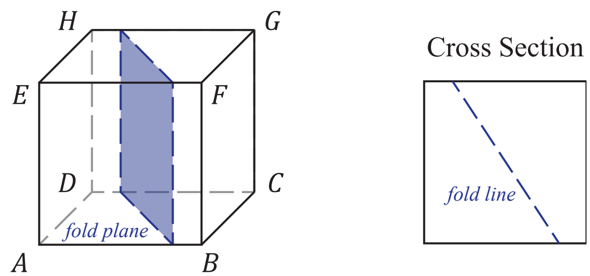


Figure 2: *Configuration of the cube with an asymmetric fold plane and the corresponding cross section.*

Thus, we choose to fix line $\overline{BC}$ and vertex A, and take lines $\overline{AD}$, $\overline{FG}$ and $\overline{HE}$ to always be straight, but not glued to the card. Additionally, we take the vertical lines $\overline{AE}$, $\overline{BF}$, $\overline{CG}$, and $\overline{DH}$ to always be straight. However, then, $\overline{AB}$, $\overline{CD}$, $\overline{EF}$ and $\overline{GH}$ must be crimped to enable the cube to fold flat, and we find a set of crease patterns to accomplish this (Figure 3). Observe the mirrored pattern between $\overline{AB}$ and $\overline{CD}$, as well as between $\overline{EF}$ and $\overline{GH}$. This helps the pattern fold more naturally when the card is closed, as the vertical segments help push down the folds of the crimps and balanced forces enable the card to fold more smoothly.

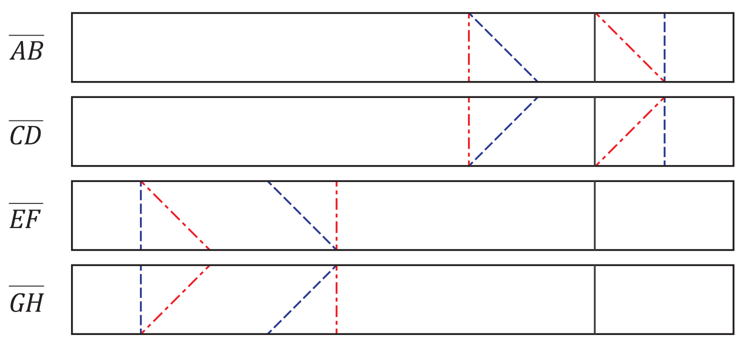


Figure 3: *Crease patterns to crimp $\overline{AB}$, $\overline{CD}$, $\overline{EF}$ and $\overline{GH}$. Dot-dashed and dashed lines indicate mountain and valley folds, respectively. Solid lines intersect the fold plane.*

With the crimps in Figure 3, the cube folds flat when the card is closed; however, we also require that the cube maintains its structure when the card is opened. In particular, since line $\overline{BC}$ and vertex A are fixed to the card, it is clear that they will assume their original position in the cube when the card is opened. However, because vertex D is not fixed to the card, and since paper is very soft, when the card is opened, vertex D may be found anywhere in the neighborhood of its original position in the cube. Hence, we use the string mechanism to pull vertex D back to its original position in the cube when the card is opened (Figure 4).

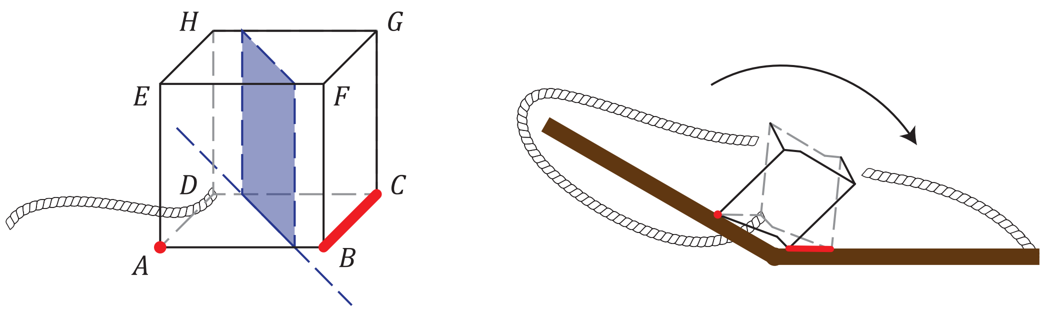
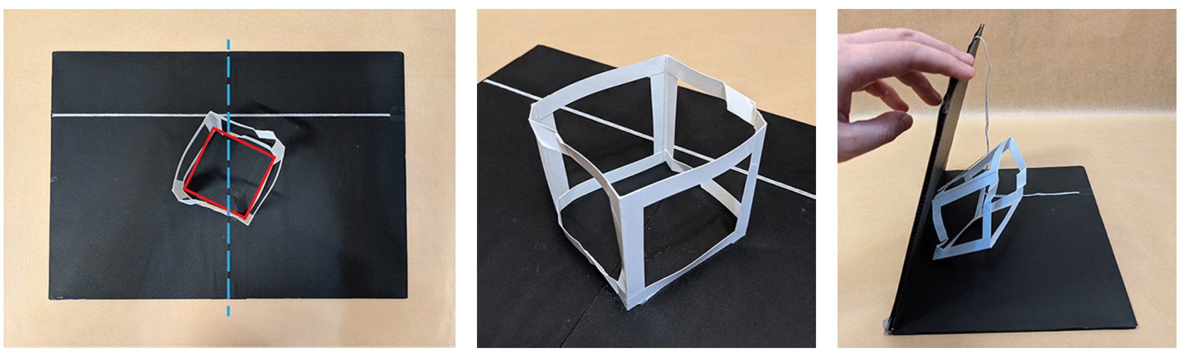


Figure 4: *Use of string mechanism for the cube. Red (bold) parts are glued to the card.*

Using 200 gsm paper, we build a pop-up card of the asymmetrically folding cube (Figures 5, 6). Indeed, we see the asymmetric fold line (Figure 5a), along with details of the cube (Figures 5b, 5c). While there may exist other designs that create the same effect, we hypothesize that an asymmetrically folding cube with solid faces may not be possible, due to the combination of crimps that would be needed on the faces. We also note that our design is not entirely asymmetric, as the fold plane in our design exhibits rotational symmetry. Creating pop-up cards of cubes with even more exotic fold planes may be a fun challenge for the future.



(a) **(b)** **(c)**

Figure 5: *Completed pop-up card from (a) a bird's-eye view, (b) an angle, and (c) partly folded.*

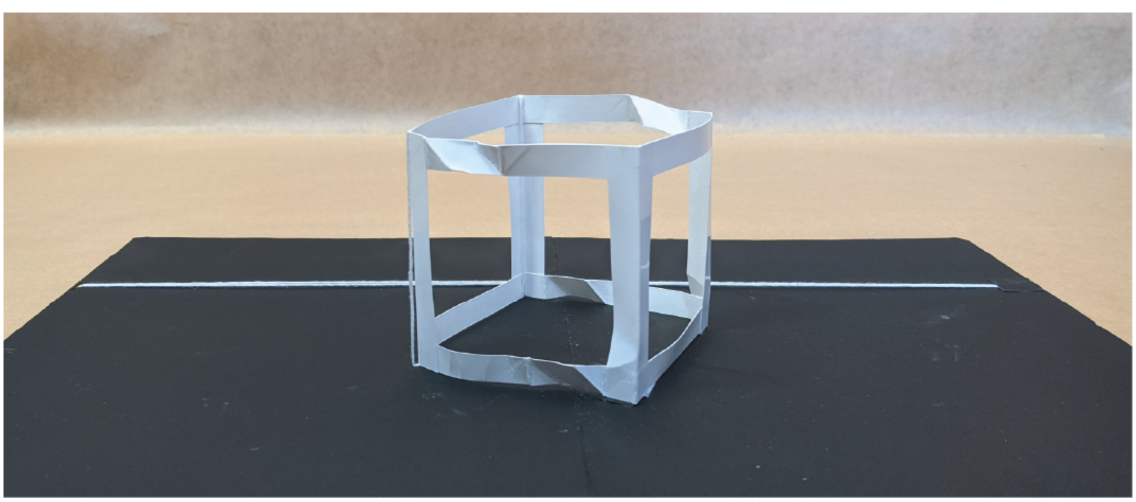


Figure 6: *A video of this pop-up card in action can be viewed at https://youtu.be/ryQHCLWLZLI.*

An Archimedean Solid

We next demonstrate the string mechanism in a pop-up card of a small rhombicuboctahedron, an Archimedean solid of equilateral triangles and squares. To enable the rhombicuboctahedron to fold flat when the card is closed, we decompose the solid into a symmetric net, a net whose left and right sides are mirror images of each other (Figure 7). By folding the net along its line of symmetry, sealing all cuts through the faces, and folding along all edges with mountain folds pointed outwards, we form a 3D solid when opposing faces are pushed together. Moreover, if the line of symmetry only cuts through one polygon, the base, we can attach it to the card, aligning the line of symmetry with the crease line so the base flattens as the card is opened.

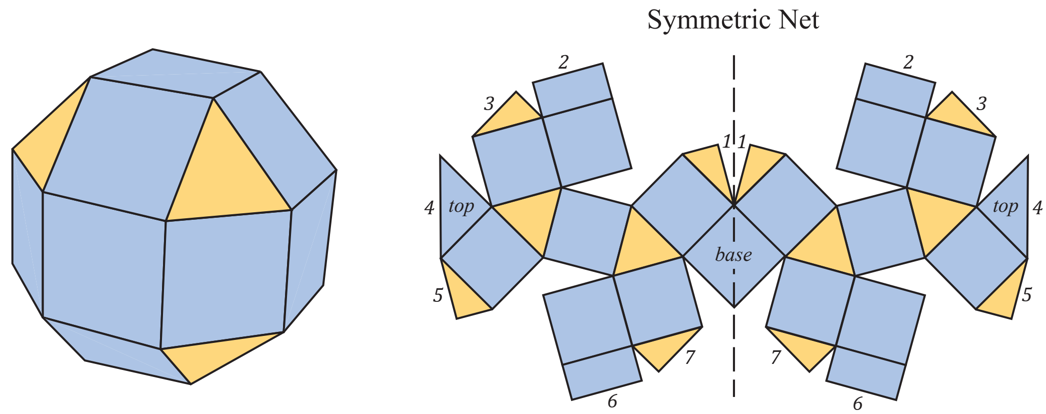


Figure 7: *A symmetric net of the small rhombicuboctahedron. Edges with matching numbers are sealed; the base and top of the solid are also indicated.*

However, since we seek to create a self-erecting pop-up card, a card where the 3D structure rises by simply opening the card, we cannot push on the sides of the net to induce the solid. Hence, we use the string mechanism, attaching one end of the string to the top square of the solid and threading it through the center of the base square and crease of the card (Figure 8). As such, when the card is opened, the top square of the solid is pulled downwards towards the card, thereby forcing the symmetric net into the 3D rhombicuboctahedron. When the card is closed, the solid collapses back into the symmetric net, enabled by the loosened string.

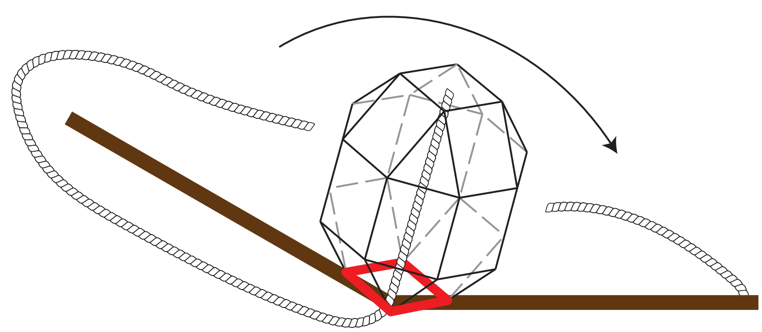


Figure 8: *String mechanism to induce rhombicuboctahedron upon opening the card. Red (bold) indicates the base, which is fixed to the card.*

Moreover, since the solid is hollow, we add an additional structure inside. In particular, we create a cherry blossom tree using a sliceform model, attach it to the base of the rhombicuboctahedron, and have both the solid and the tree fold and rise together. Since the tree must fit within the solid, it is subject to size restrictions. That is, for a rhombicuboctahedron with side length s, the size of the tree at its longest and widest points cannot exceed $(1+\sqrt{2})s-\varepsilon$ for some $\varepsilon > 0$ (Figure 9). In practice, ε is non-negligible for aesthetic purposes.

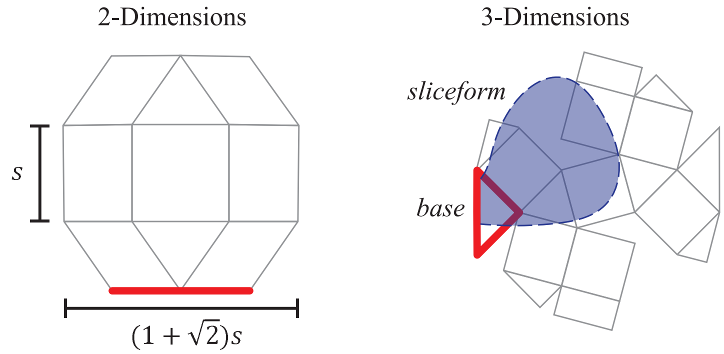


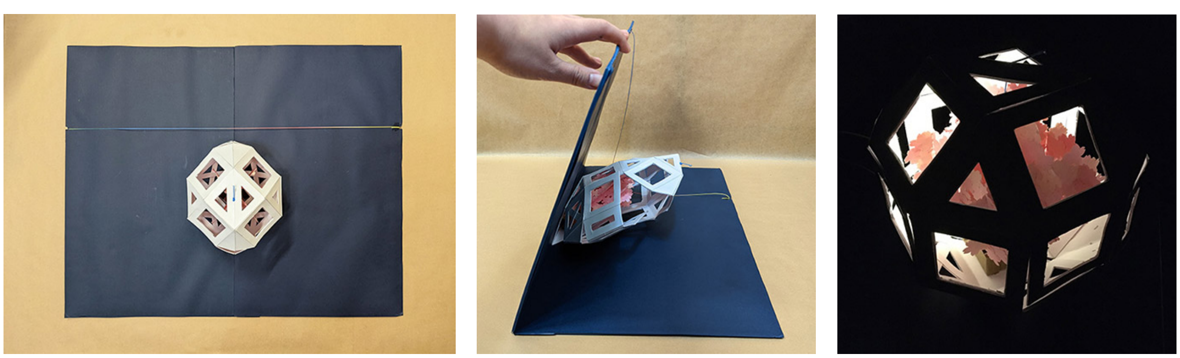
Figure 9: *Size restrictions of the tree for a rhombicuboctahedron with side length s, with a rough measure of the 2D tree relative to the symmetric net. Red (bold) indicates the base, which is fixed to the card.*

We assemble the sliceform cherry blossom tree from over 200 individual petals (Figure 10). In particular, the sliceform adheres to the size restrictions set by the rhombicuboctahedron in 2D (Figure 10a) and in 3D (Figures 10b, 10c). Note that when collapsed, the tree is at a 45° angle relative to the base of the solid.

(a) **(b)** **(c)**

Figure 10: *Sliceform cherry blossom tree (a) in 2D, (b) in 3D, and (c) in the solid.*

Finally, we build a pop-up card of the rhombicuboctahedron (Figures 11, 12), with each face of the solid reinforced with five layers of 200 gsm paper. Upon assembly, we see that the sliceform aligns with the solid (Figure 11a) and that the structures fold synchronously (Figure 11b). Additionally, we can add a small light bulb in the solid to turn the card into a lantern (Figure 11c).



(a) **(b)** **(c)**

Figure 11: *Completed pop-up card from (a) a bird's-eye view, (b) partly folded, and (c) as a lantern.*

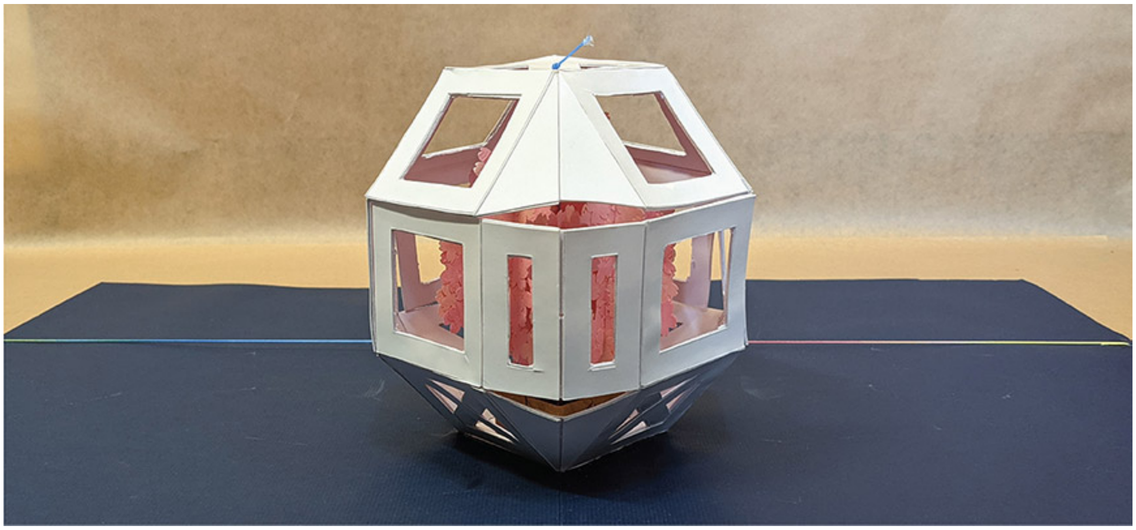


Figure 12: *A video of this pop-up card in action can be viewed at https://youtu.be/SQLPqibB-z0. While the video shows the opened card being held down, the structure can indeed be freestanding.*

Thus, we demonstrate a design for a pop-up card of an Archimedean solid, the small rhombicuboctahedron, showing the usefulness of our string mechanism in creating polyhedral pop-up cards. Incorporating more strings, that is, a system of pulleys, in future designs may enable the solid to close more tightly.

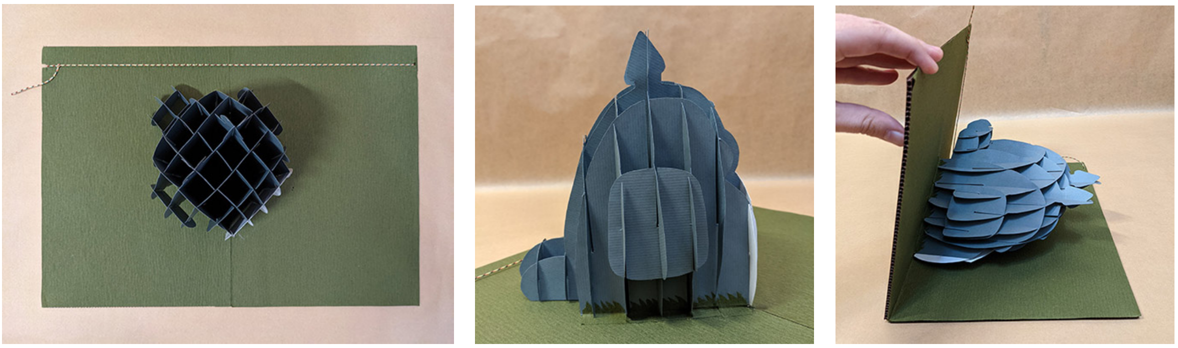
Symmetric nets for polyhedra

In theory, it should be possible to create pop-up cards in this same manner for other polyhedra, provided the solid has a symmetric net and that the line of symmetry cuts through only one face of the solid, which is the face attached to the card. Under such criteria, we find another symmetric net for the rhombicuboctahedron [7] and symmetric nets for the deltoidal icositetrahedron [14] and the disdyakis dodecahedron [15]. We hypothesize that pop-up cards for these solids can be built using these symmetric nets. Accordingly, an open problem would be to design pop-up cards for other Platonic, Archimedean, and Catalan solids using symmetric nets. For solids with more than one symmetric net, it may be interesting to compare the effectiveness of each net for pop-up card design, as the force the string mechanism exerts on the solid when the card is opened will differ between nets. More fundamentally, to the best of our knowledge, it remains to be shown whether every Archimedean and Catalan solid has a symmetric net, which is a nice theoretical challenge.

The study of symmetric nets for polyhedra under various criteria is an important research topic—not only helpful for designing pop-up cards, but also manifesting itself in other engineering applications. In robotics, foldable polyhedral nets are studied for non-intersecting motion and for the design of self-assembling systems. Previous studies cite the compactness of nets as a key indicator for maximizing the yield of surface-tension driven self-folding polyhedra [11]. To these ends, we hypothesize that the symmetry of nets could also influence yield and might be an interesting design criterion to investigate further.

A Sliceform Example

Finally, we show our string mechanism also applies to sliceform-based pop-up cards. A sliceform consists of a series of intersecting slices, which create a movable lattice; the slices act as hinges, thereby enabling the pop-up to rise and collapse [10, 13]. Here, we create an original sliceform model of the main character from the 1988 film *My Neighbor Totoro* (Figures 13, 14). We orient the sliceform such that the grid formed by the slices is 45° relative to the crease of the card. With one outer corner of the sliceform on the crease line, we glue one outer edge to the card and attach the vertex on the opposing side to the loose end of the string. As such, when the card is opened, the corner is pulled to its desired position; when the card is closed, the corner moves freely, thereby collapsing the sliceform. Indeed, we see the grid of the sliceform (Figure 13a), the attached edge (Figure 13b), and the collapsed sliceform (Figure 13c). Thus, our string mechanism can also be applied in the design of sliceform-based pop-up cards.



(a) **(b)** **(c)**

Figure 13: *Completed pop-up card from (a) a bird's-eye view, (b) an angle, and (c) partly folded.*

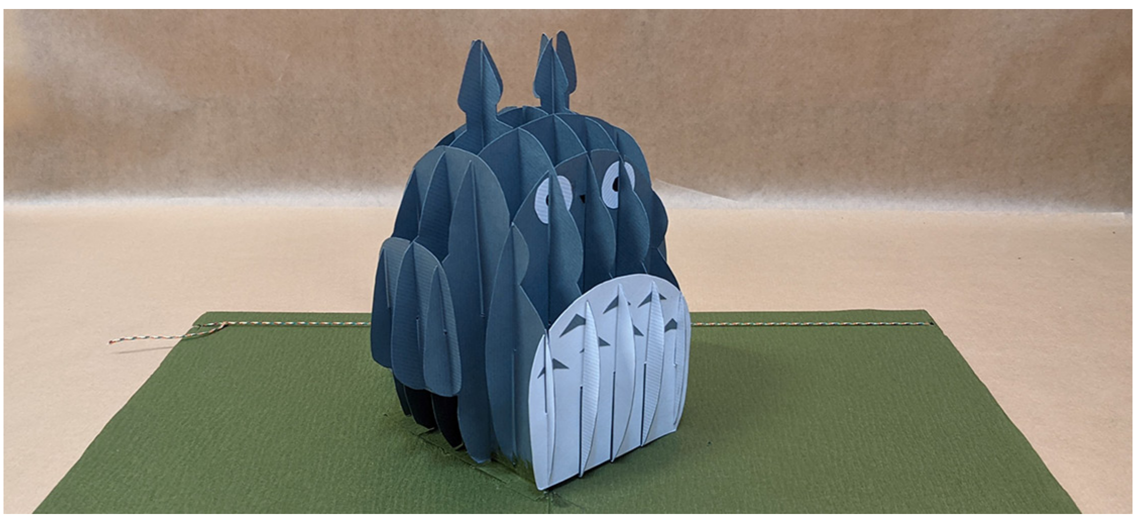


Figure 14: *A video of this pop-up card in action can be viewed at https://youtu.be/LVmLLI_S6NE.*

Summary and Conclusions

In this paper, we present a novel string mechanism useful for inducing the 3D structure of 180° pop-up cards. Using this technique, we exhibit two new polyhedral pop-up card designs: an asymmetrically folding cube and a rhombicuboctahedron. Additionally, we create an original sliceform pop-up card to show our mechanism can be used in tandem with other, more widespread, design techniques. In all the designs, the string mechanism enables the pop-up element to form a rigid structure when the card is opened, while still allowing the card to fold flat when closed. There does exist pop-up cards that use elastics in a similar manner [8]. However, in this case, the elastic is stretched when the card is folded, and hence, the card requires external intervention, such as an envelope, to stay folded. In contrast, the string mechanism we present is stable when the card is both opened and closed.

We also address symmetric nets for polyhedra. Indeed, we conjecture on other possible pop-up card designs for polyhedra, having found symmetric nets for other solids. Additionally, we discuss open problems relating to symmetric nets, extending beyond the scope of pop-up cards and into the realm of robotics and engineering. Overall, the ideas we present in this paper may aid in the design of more sophisticated and magical pop-up cards, as well as spark new avenues of research relating to polyhedra and its complexities.

Acknowledgements

The author would like to thank Erik Demaine and Martin Demaine for all their help and guidance. This work started as a final project in an MIT class on Geometric Folding Algorithms (6.849, Fall 2020).

References

[1] Z. R. Abel, E. D. Demaine, M. L. Demaine, S. C. Eisenstat, A. Lubiw, A. Schulz, D. L. Souvaine, G. Viglietta, and A. Winslow. "Algorithms for designing pop-up cards." *Proceedings of the 30th International Symposium on Theoretical Aspects of Computer Science*, pp. 269–280, 2013.

[2] M. Benedict. *Pop-Up Twirl Card*. 2010. https://www.splitcoaststampers.com/resources/tutorials/popuptwirl/.

[3] M. Chatani and K. Nakazawa. *Pop-Up Geometric Origami*. Ondorisha, 1994.

[4] P. Dahmen. *12 Pop-Up Paper Sculptures*. 2019. https://www.youtube.com/watch?v=05FEzSCBGoM&t=2s.

[5] M. Damian, R. Flatland, and J. O'Rourke. "Grid Vertex-Unfolding Orthogonal Polyhedra." *Discrete & Computational Geometry*, vol. 39, 2008, pp. 213–238.

[6] E. D. Demaine, D. Eppstein, J. Erickson, G. W. Hart, and J. O'Rourke. "Vertex-unfoldings of simplicial manifolds." *Proceedings of the eighteenth annual symposium on Computational geometry*, pp. 237–243, 2002.

[7] S. L. Gould. "Using Geometer's Sketchpad to Construct Pop-up Polyhedra as a Tool for Classroom Study of Geometry." *Bridges Conference Proceedings*, Banff, Canada, Jul. 26–30, 2009, pp. 351–354. http://archive.bridgesmathart.org/2009/bridges2009-351.pdf.

[8] T. Laughlin. *11 Amazing Pop-up Cards - rubber band activated!*. 2009. https://www.youtube.com/watch?v=vYUx0zPn2g0.

[9] Lego®. *Ideas Pop-up Book (Jack & the beanstalk)*. https://www.lego.com/en-us/product/pop-up-book-21315.

[10] M. G. Monera. "Cardboard Construction of the Sphere by the Stereographic Projection." *Bridges Conference Proceedings*, Aug. 1–5, 2020, pp. 169–174. http://archive.bridgesmathart.org/2020/bridges2020-169.pdf.

[11] S. Pandey, M. Ewing, A. Kunas, N. Nguyen, D. H. Gracias, and G. Menon. "Algorithmic design of self-folding polyhedra." *Proceedings of the National Academy of Sciences of the United States of America*, vol. 108, no. 50, 2011, pp. 19885–19890.

[12] Rich. *Valentines Day Pop Up Card: 3D Heart*. 2011. https://www.youtube.com/watch?v=F2li2WrS8gQ&feature=emb_imp_woyt.

[13] J. Sharp. "Sliceform Sculptures-a Bridge between Art and Mathematics." *Bridges Conference Proceedings*, Winfield, USA, Jul. 28–30, 1998, pp. 275–276. https://archive.bridgesmathart.org/1998/bridges1998-275.pdf.

[14] E. W. Weisstein. *Deltoidal Icositetrahedron*. MathWorld–A Wolfram Resource. https://mathworld.wolfram.com/DeltoidalIcositetrahedron.html.

[15] E. W. Weisstein. *Disdyakis Dodecahedron*. MathWorld–A Wolfram Resource. https://mathworld.wolfram.com/DisdyakisDodecahedron.html.