Quadrilateral Spiral Tilings and Escheresque Art

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Abstract

When a quadrilateral with no two opposite edges equal in length, is extended along its two shorter edges by copying a scaled copy of the same over and over, they converge to a single point in plane. However, in general, the quadrilaterals do not coincide when the clockwise and counter-clockwise arms meet. This paper searches for the conditions under which such coincidence is possible and hence provides a method to construct such spirals. The paper further explores how these quadrilateral spirals can be used to create Escheresque tessellations and convert some of Escher's planar tilings into spiral ones.

Introduction

In 1991 David Henderson posed a problem [6]. Draw a quadrilateral with no two opposite edges equal in length. Choose a pair of opposite sides, and on the shorter of the two, attach a (reduced) scaled copy of the quadrilateral. Keep doing this over and over. Do the same for the other pair of opposite edges. The two 'arms' thus created converge to a single point in plane. While this is true for all quadrilaterals, as explained by James Tanton in his monthly post [7], the two arms do not necessarily coincide when they first overlap. Vincent Pantal has produced a beautiful GeoGebra applet to visualize this [3]. Robert Fathauer has, very well, described them in his recent book [4]. Dániel Erdély also developed a triangle spiral system called spidrons [5]. This paper will focus on conditions required for a given quadrilateral to form clockwise and counter-clockwise spirals such that they coincide at every intersection. The paper further explores, with examples, how the derived math can be used to make Escheresque tessellations and adapts some of Escher's planar divisions, into spiral tiling. Figure 1 shows an example in which the 6th unit of counter-clockwise spiral does not coincide with 8th unit of clockwise spiral. This paper algebraically explores the method to obtain the desired result as shown in Figure 2.

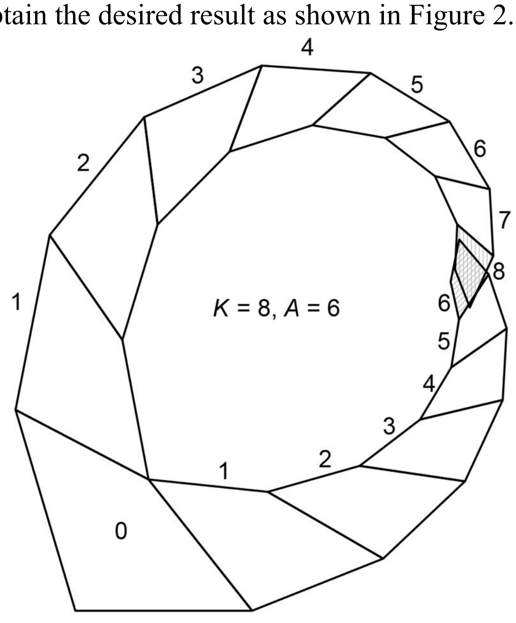


Figure 1: *Quadrilateral spirals not coinciding.*

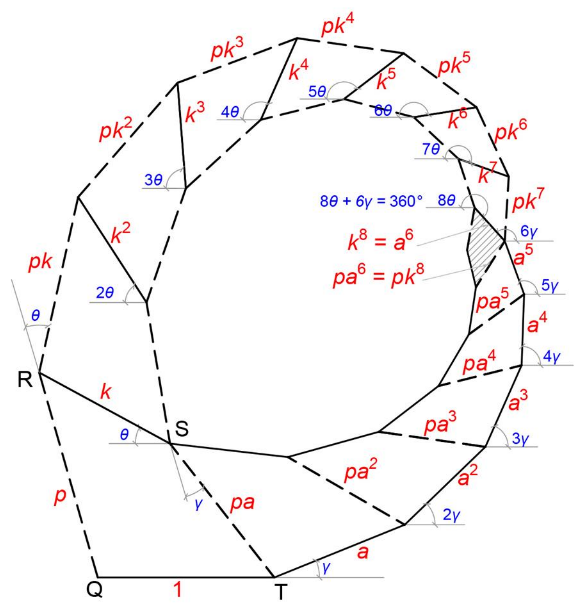


Figure 2: *Quadrilateral spirals coinciding.*

Equations based on Geometry of Quadrilateral Spirals

Based on the geometry of quadrilateral spirals, equations in terms of its variables are derived. In Figure 3, consider a quadrilateral QRST with point Q on the origin. Let l(QT) = 1 and l(QR) = *p*. Let *k* and *a* be the factor by which the opposite sides of the quadrilateral are reduced. Thus l(RS) = *k* and l(ST) = *ap*. Let γ and θ be the angle an adjacent quadrilateral makes with a previous quadrilateral, or in other words with $\overline{QT}$ and $\overline{QR}$ respectively. Let $\overline{QR}$ make angle β with the y–axis. In the spirals we will consider that *K* and *A* are the number of quadrilaterals required in clockwise and counter-clockwise direction respectively, before they coincide.

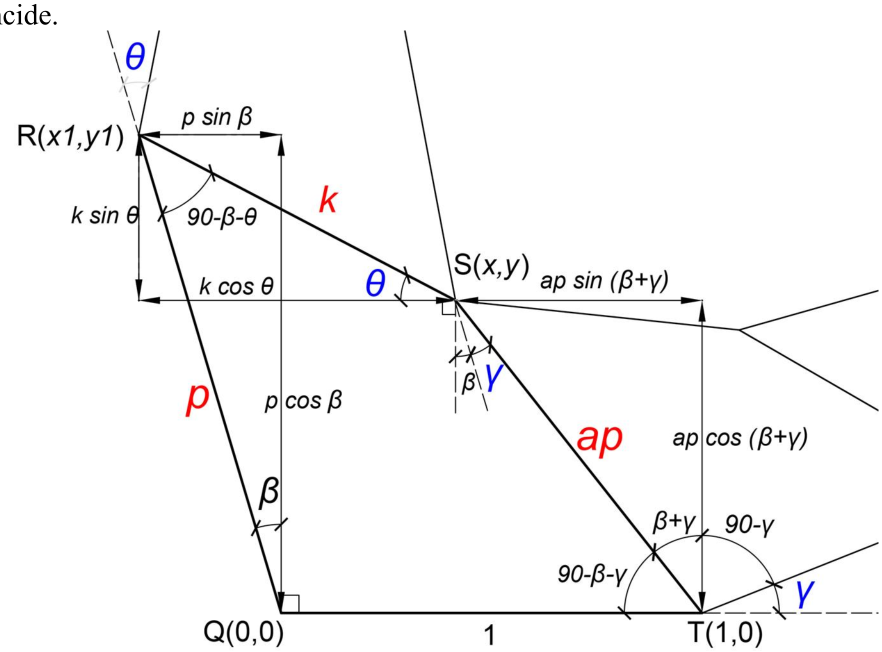


Figure 3: *Variables assigned to the first quadrilateral.*

For a given value of *K*, *A*, γ and *k*, we will calculate the values of θ, *a*, *p* and β. Once we know all these variables, the shape of the quadrilateral can be determined. In Figure 2 we can observe that $\overline{QT}$ is rotated by γ and scaled by factor of *a,* and the chain is repeated in the counter-clockwise direction *A* times; this segment is also rotated by θ and scaled by a factor of *k* (to form $\overline{RS}$) in the clockwise direction and then repeated K times. With these conditions we can establish following equations:

$$K\theta + A\gamma = 360°. \text{ Therefore } \theta = \frac{360 - A\gamma}{K}. (1)$$

$$k^K = a^A. \text{ Therefore } a = \sqrt[A]{k^K}. (2)$$

Similarly $\overline{QR}$ is rotated by θ and scaled by factor of *k,* and the chain is repeated in the clockwise direction *K* times; the segment is also rotated by γ and scaled by a factor of *a* (to form $\overline{TS}$) in the counter-clockwise direction and then repeated *A* times to form the opposite sides of the quadrilateral. It should be noted that with these conditions we can establish exactly the same equations as (1) and (2). We can therefore establish that when both these equations are satisfied, the quadrilaterals will coincide. In Figure 3 (a zoomed-in diagram of Figure 2), the co-ordinates of point *S* can be expressed as follows:

$$x = 1 - ap\ sin\,(\beta + \gamma). \quad (3)$$

$$y = apcos\,(\beta + \gamma). \quad (4)$$

The co-ordinates of point *R* can be expressed in two ways:

$$x1 = x - k\ cos\theta\ ;\ x1 =- p\ sin\beta. \quad (5)$$

$$y1 = y + k\ sin\theta;\ y1 = p\ cos\beta. \quad (6)$$

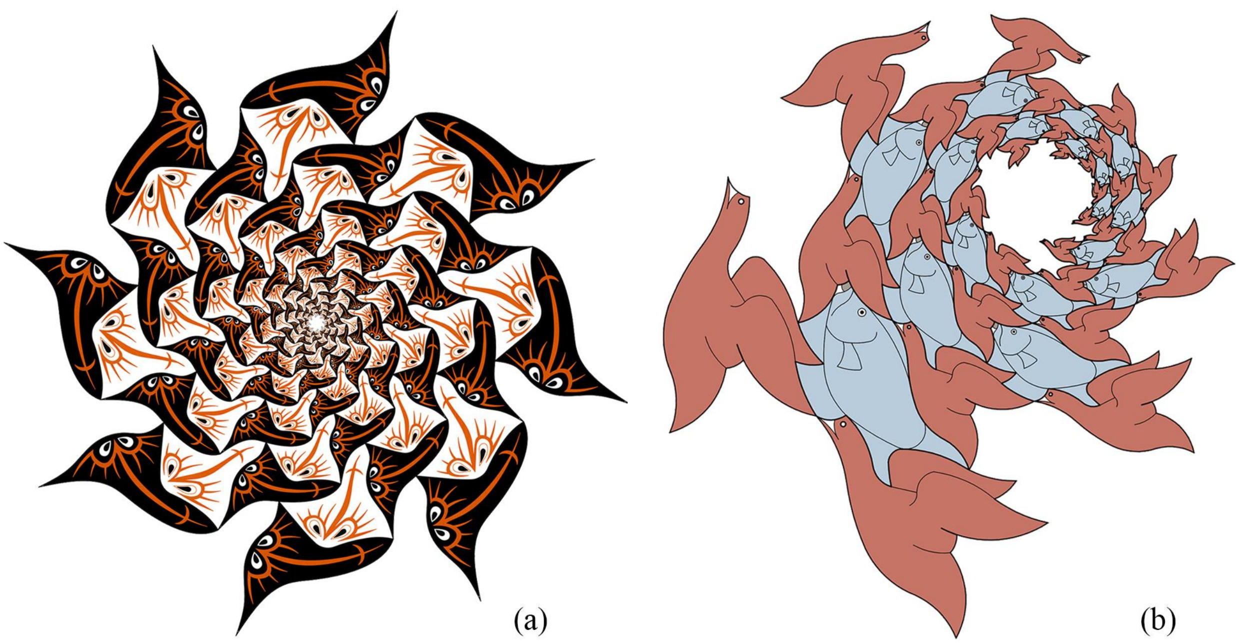
By solving the equations (3) up to (6) simultaneously it can be show that the value of variable:

$$\beta = \tan^{-1}\left[\frac{m - ma\cos\gamma - a\sin\gamma}{a\cos\gamma - 1 - ma\sin\gamma}\right], \text{where } \left(m = \frac{1 - k\cos\theta}{k\sin\theta}\right) \quad (7)$$

We can now construct the desired quadrilateral as all its coordinates: Q, R, S and T are known. The above equations are plotted on the graphing calculator *Desmos.* The following input variables are controlled by a sliding bar: A, K, γ and k. This file is freely available and users are encouraged to play with variables in order to obtain the desired shape for producing new spiral designs [1].

Spiral versions of Escher's Original Art

Most of M.C. Escher's tessellations were periodic and classified according to the type of polygon in the underlying geometric lattice. Letter *A* denoted an arbitrary parallelogram, *B* any rhombus, *C* any rectangle, *D* a square, and *E* an isosceles right triangle [2]. One could now add the letter *F* to his system assigned to an irregular quadrilateral. For Escher, regular division of the plane was a means to capture infinity [2]. He also produced some prints using hyperbolic and spiral geometry like "Circle Limit I" (1958) and "Path of life I" (1958), which have rotational symmetry. But we are not aware of any print of his which has an asymmetric spiral. Equipped with the formulas derived previously it is possible to produce Escheresque designs using irregular quadrilaterals. Since the number of spirals, shape of the quadrilateral and internal angles can be chosen at the artist's discretion, the possibilities are endless, and an array of designs can be produced. Some examples of such artworks are shown in Figures 4 and 5.



(a) (b)

Figure 4: *Examples of some spiral versions of Escher's original artwork, produced by the author.*

Figure 4a is a spiral version of Escher's "Path of Life, I" (1958). Escher's print had eight stingrays swimming in ever decreasing circle, but the illustration lacked a spiral impression. In this version one can see the slight angular rotation every time the infinite circle decreases in size. Figure 4b is a spiral version of Escher's Bird Fish (1938). In this design the birds and fishes alternate between cells. While the number of units are the same in both directions (eight in this case) it should be noted that the shape of the cell itself is based on an irregular quadrilateral, and not a kite. Figure 5a is the spiral versions of Escher's Regular Division of Planes with Birds (1949). In this design the 7th counter-clockwise bird coincides with the 9th clockwise bird. It is noted that due to the asymmetric nature of the spirals, no two motifs are of the same size. Figure 5b utilises Escher's Drawing No.134 (flowers) with two square tiles. Here two quadrilateral tiles get tessellated in spiral fashion to produce the overall illustration.

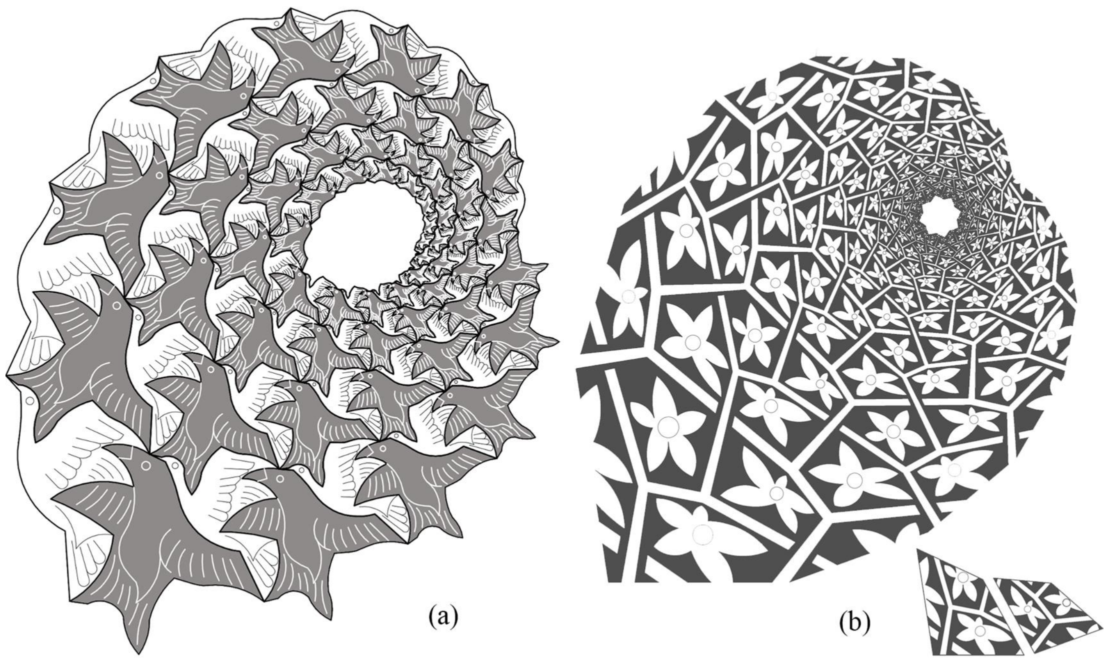


Figure 5: *Examples of spiral versions of Escher's original artwork, produced by the author.*

Escheresque art using Spiral Geometry

In Figures 4 and 5 we saw motifs, designed by Escher, transformed into spiral designs. In Figures 6 up to 9, we will see some new motifs employed for the same purpose. In Figures 6a, 6b and 7a we use concave quadrilateral as a module to produce the illustrations. The underlying grid for these uses spirals formed by the variables $K = 4$, $A = 1$, $\gamma = 72$ and $k = 0.5$. Figure 7b uses a triangle, as a module, formed when any one angle of the quadrilateral is 180° .

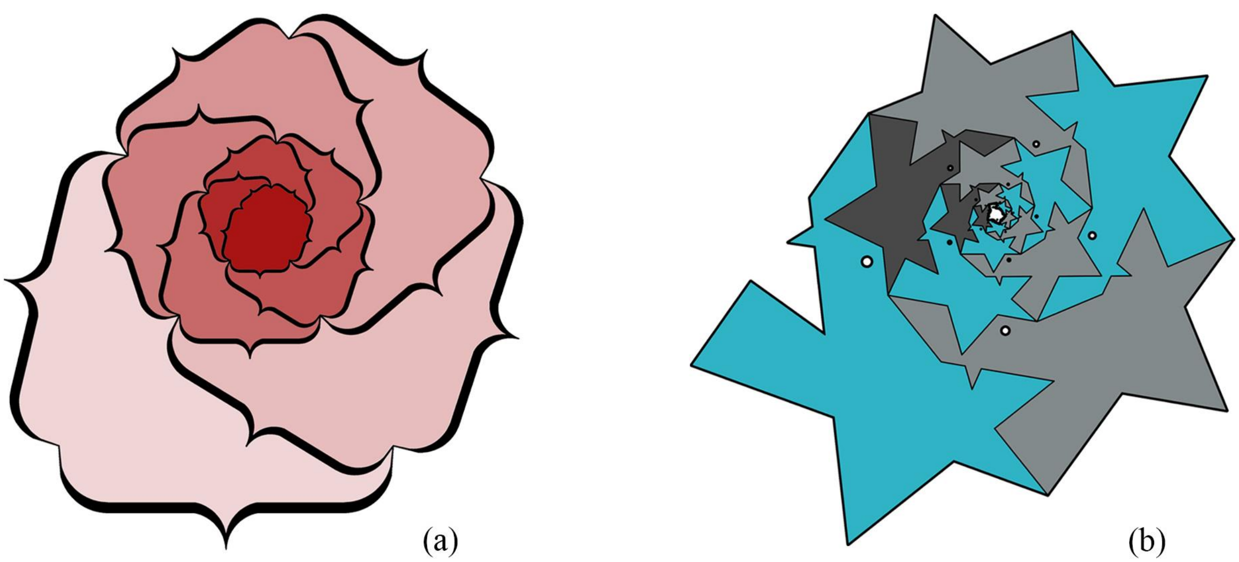


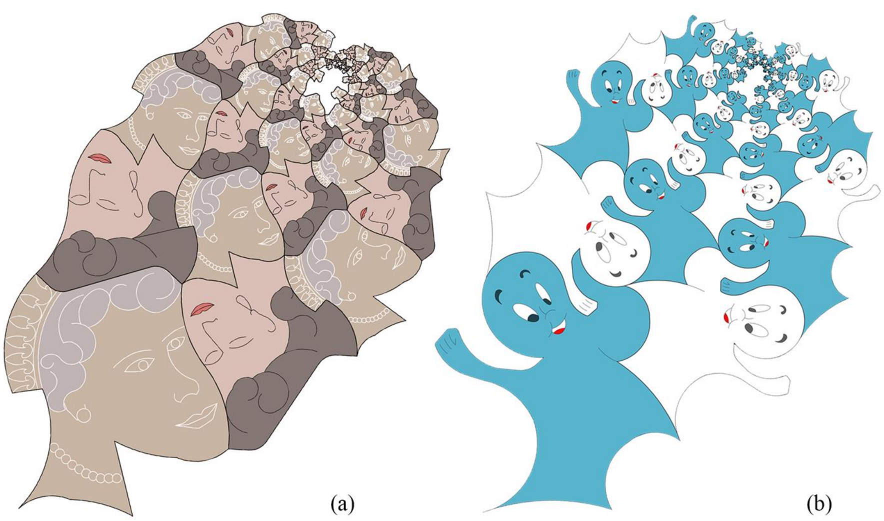
Figure 6: *Examples of spiral versions of Escher's original artwork, produced by the author.*

Figure 6a is called the "Curly-bracket Rose". Each side of the quadrilateral is replaced by a curly bracket leading to a graphic resembling a rose. Figure 6b is named "Biting sharks" as it elicits graphical sharks biting each other's tail; five in counter-clockwise and two in clockwise direction. Figure 7a is named "Devil Moon." One can either see a laughing crescent moon or a smiling devil because the devil's horn can also be perceived as a smile. Figure 7b is called "Wardrobe of Escher's girlfriend." In this graphic asymmetric spirals made from stilettos tessellate to fill the 'wardrobe' plane.

(a) (b)

Figure 7: *Examples of Escheresque artwork, by the author, with irregular quadrilaterals module.*

Escher had classified his tessellations into 5 "groups" which corresponded with the plane symmetry groups as follows: *p1, p2, pg, pgg, p4* [2]. If the underlying quadrilaterals of the motifs in Figures 4 through 7 were parallelograms, the tilings would have symmetry group p1. It is noted that the other 4 groups are not possible with an irregular quadrilateral, however by choosing the shape of quadrilateral close to either a rhombus, rectangle or square, one can produce tessellations of the remaining 4 "groups." Examples of these produced by the author are shown in Figure 8 and 9. In Figure 8a, named "Young and Old Queen", if all of the motifs were congruent, the tiling would have symmetry group p2. Figure 8b is called "Casper's wave" and if all of the motifs were congruent, this would be a pg tiling. Figure 9a is an abstract pattern, and if all of the motifs were congruent, it would have pgg symmetry group. Figure 9b is called "Dancing Dogs, and if all of the dog motifs were congruent, it would be a p4 tiling. It should be noted that the shapes shown here are not similar but almost similar and this method could be practically employed for the purpose of art.



(a) (b)

Figure 8: *Examples of Escheresque artwork based on plane symmetry groups: (a) p2, (b) pg.*

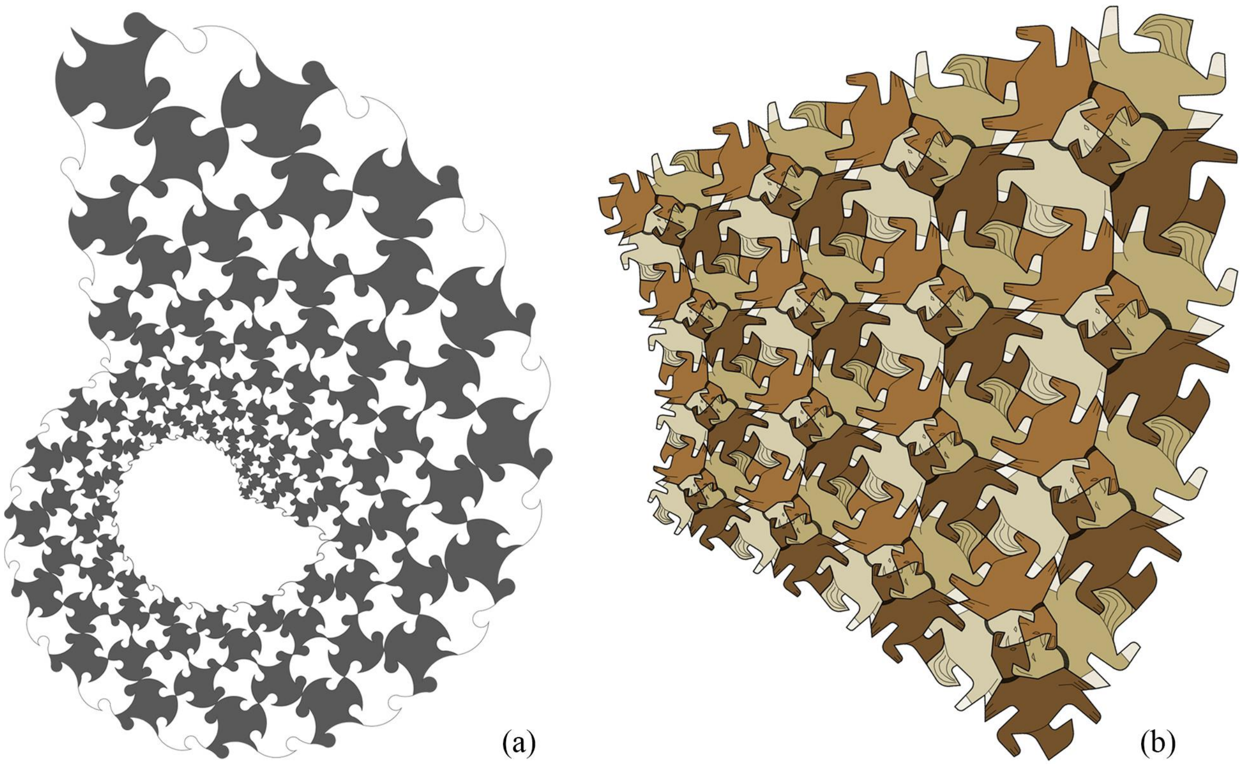


Figure 9: *Examples of Escheresque artwork based on plane symmetry groups: (a) pgg, (b) p4.*

Conclusion

It's possible to draw intersecting quadrilateral spirals and the geometry can be explored in Escheresque tessellations. The method could be used in various other art forms, architecture, folded plate structures, origami, graphic design, puzzles, and product design. We hope that more creative artworks and applications will be explored using this method of producing spiral quadrilaterals. The freely available file should be used to determine the unit shape of quadrilateral and its accurate coordinates used to draw them out.

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