Infinitely Invertible Infinity

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Abstract

We present our experiences crafting cloth objects with intriguing inversion properties and tell a story about how the design aspirations of a seamstress can drive a mathematical inquiry.

Introduction

If you haven't spent much time contemplating what happens when you turn things inside out, we think you've been missing out. Of course, turning cloth objects inside out is something people do routinely, especially when folding laundry…so perhaps it doesn't strike you as a topic that is particularly mysterious or interesting. For example, it's not hard to predict what will happen when you invert a standard pillowcase. It looks pretty much the same afterwards, except the inside surface fabric becomes the outside surface, and vice versa. But suppose you have a donut-shaped seat cushion. And further suppose, as shown in Figure 1, it is constructed with a removable foam cylinder inside a zippered donut-shaped pillowcase, so that you can remove the foam and wash the pillowcase separately. Aside from the change in exterior surface fabric, will this pillowcase look the same if it gets accidentally turned inside out in the wash? If you are a topologist, you might assert that the answer is "Yes, it will look the same." But if you are a geometer, or perhaps a seamstress, you'll likely have to concede that oftentimes the answer is an emphatic "No, it will not look the same at all! "

Our explorations began with just such a question, which led to a variety of related inversion questions. This paper is about that process, the design aspirations that motivated us, and the engaging sequence of puzzles that emerged from it. We discuss what we learned along the way and how it enabled us to produce a series of cloth objects (some of them wearable art) with intriguing and non-intuitive inversion properties.

cylindrical foam stuffer for the pillowcase

zipper slit for removing the foam cylinder
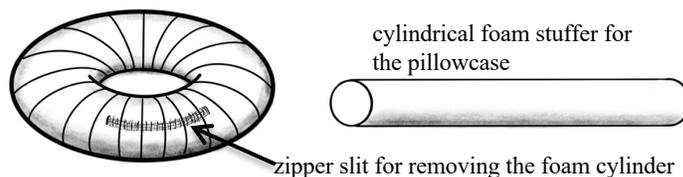
Figure 1: *A toroidal pillowcase with cylindrical foam insert.*

Inversion Woes: A First Encounter

Our first experiments began several years ago with what seemed like a simple project – sewing an "infinity" scarf like the commercially available one shown in Figure 2a. For our purposes (and sometimes in fashion lingo) an infinity scarf is defined as a hollow cloth torus, similar to the toroidal pillowcase in Figure 1, but unstuffed and whose hole is large enough to fit over the head so that it can be looped around a neck one or more times. The mathematical definition of a topological torus suggests a method for sewing such a scarf from a rectangular piece of fabric by sewing together its opposite edges (Figure 3). Initially, all we wanted was to stitch it together on a sewing machine in such a way as to hide the raw seam edges on the interior.

To accomplish this, we tried a standard sewing strategy: stitch the desired object with the "back" side of the fabric facing outward, but with a small slit left in one seam, through which it is later inverted to hide the raw seams on the interior (Figure 2b left). But the strategy failed! When inverted to expose its "front" side, our scarf's shape looked mystifyingly wrong and entirely unwearable (Figure 2b right). The question about what went awry and how to correctly execute this strategy on an infinity scarf sewn from a rectangular piece of fabric was later published as a puzzle in Gary Antonick's New York Times Numberplay blog [3].

With our initial tangible failure in hand, the mathematics behind the problem became clearer: When a torus is inverted, its meridians (rings going *through* the torus hole) and longitudes (rings going *around* the torus hole) swap roles, as shown in Figure 4. Reimagining Figure 3 such that the red edges (instead of the blue) are sewn together first makes it clear how either set of opposite edges can play either role. The sewing strategy –in this case the choice of initial sewing seam –determines which is which…at least until you invert it. Additional discussion is provided in the blog post[3], but the crux of the solution is simply to sew it inside out in its strange "wrong" form so that, once inverted into its "right" form, it will look and function as desired with the raw seams hidden on the interior.

Figure 2: *(a) A commercial infinity scarf by Lily Pulitzer. (b) A trial doll-sized scarf showing what went wrong when sewn inside out (left) and then inverted (right) to hide the raw seam edges.*
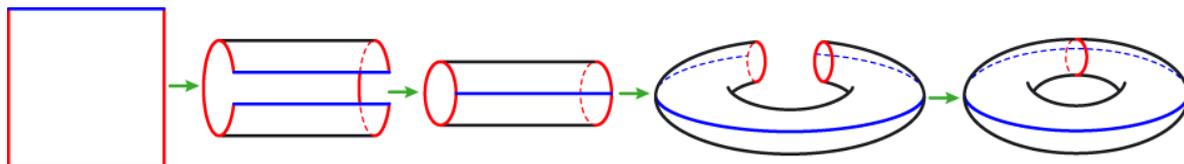
Figure 3: *A topological torus: a stretchy shrinkable square whose opposite edges are understood to be connected.*

Figure 4: *Stages of a punctured, stretchy, shrinkable torus being turned inside out. These are still images from an online animated gif by Surot [1] showing how the meridians (circles going through the torus' hole) become longitudes (circles going around the torus' hole) on inversion, and vice versa.*

Another Inversion Lament

The solution to the seam-hiding problem was satisfying and effective, but the experience gave rise to a new lament. At the fabric store, we'd noticed some lovely reversible fabrics with different colors and patterns on each side -- how sad that it did not seem viable to use them to make a fully reversible toroidal scarf that could be worn with either surface showing. Of course, this would create a new issue about where to hide the seam flaps, but the bigger problem –which felt like a deal-breaker –was the shape change on inversion. We contemplated a scarf made from a square layout, which would preserve its shape when inverted, but rejected that as lacking proper proportions for an infinity scarf. Was a non-square reversible infinity scarf really impossible?

What we discovered in pondering this question is that there is one other toroidal geometry that inverts to the same shape. Described in detail in our Bridges 2017 paper [5], it can be made by sewing together the opposite edges of a hexagon. Although proportionally close to a square, this hexagonal layout turns out to have some useful and interesting properties that make it more suitable for a toroidal scarf, such as a Moebius-like twist (which makes a scarf with a single loop drape nicely on a human figure) and the ability to fold flat into a six-layer equilateral triangle (which makes it store neatly and compactly). Using it, we created something new that we'd never seen before – a truly functional invertible infinity scarf! (All references to "reversible infinity scarves" that we have found turn out to be two-sided bands whose different sides were both potentially visible when worn. By "invertible infinity scarf" we mean a cloth torus whose alternative surface is revealed by turning it inside out.) Sewn with custom fabric designed to echo the hexagonal structure of the scarf so that the printed patterns match up at the seams, it was presented in the mathematical art shows at Bridges and JMM in 2017 (Fig. 5).

Figure 5: *A rendition of the hexagonal scarf,* "*Invertible Infinity,*" *shown at the Joint Mathematics Meetings 2017 exhibit of mathematical art. The "filmstrip" at the bottom shows the inversion progressing from one equilateral triangle folding to the other.*

Those Pesky Seams

Despite these successes, one aspect of our design remained aggravating. With its hexagonal structure, both inversions of the scarf were now functional, but this also meant we'd lost the hiding place for the raw seam edges. As a workaround for our first reversible scarves like the one in Figure 5, we used French seams, which conceal the ragged edge but still leave a visible flap on one side. While tidier looking than a raw-edged flap, this wasn't as neat as a standard hidden seam and it also disrupted the perfect symmetry on inversion that we aspired to. The problem could be mitigated somewhat with delicate hand-stitching to sew the flaps flat or to otherwise minimize the appearance of excess fabric on one side. But that required tricky

extra construction work and still violated our symmetry-seeking aesthetic. However, after sewing a number of French seams, a new possibility emerged.

Since the available options in stores for reversible fabrics were limited, and because our pattern design aspirations for them became more specific and ambitious, we began creating our own reversible fabrics by stitching together two custom printed fabric layers back to back. At first the extra bulk of a double layer seemed disadvantageous over the store-bought reversible fabrics, in which the front and back patterns are woven together into a single layer. However, especially with lightweight fabrics, it worked well…and at some point we noticed a promising hypothetical. Between our two layers, where the wrong sides of the fabric touched, was a hidden region that didn't exist in single-layer reversible fabric. So, at least in principle, there was an unused space in which to conceal the seam flaps –if only we could figure out how to access it with a sewing machine. (We jokingly began referring to this hidden region as the "inner sanctum.")

For a more traditional planar rectangular scarf (i.e., the kind typically worn by snowmen), there is an easy solution for constructing a reversible two-layer version with hidden seams. Figure 6a shows a straightforward approach in which two long rectangular "pillowcase" shapes are sewn separately and one is then stuffed inside the other, back-to-back (i.e., seam-flap sides together) to nest them. This method was used to create the reversible Superman/Clark-Kent design with hidden seams shown in Figure 6c. However, it isn't obvious how to translate this approach to a toroidal scarf (Figure 6b) because you can't nest two tori simply by stuffing one inside a slit in the other. If you try, rather than nesting properly and looping around the hole, the inner torus winds up scrunched in a ball on one side of the interior tunnel of the outer torus.
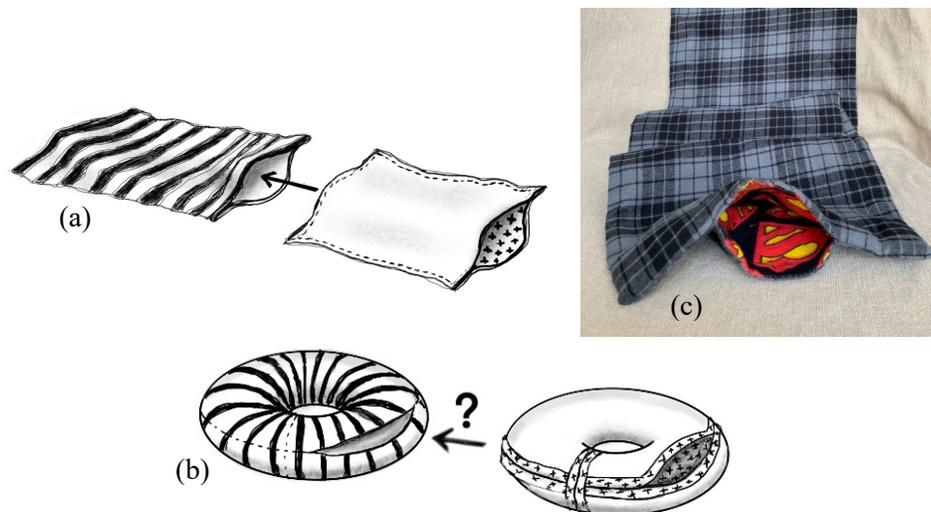
Figure 6: *A nesting method (stuffing one scarf inside a slit in the other) that works if the two scarves are topological spheres (a), but not if they are tori (b). This method was used to create the reversible scarf with hidden seams for Superman/Clark Kent shown in (c).*

There is more than one approach to this problem, but we found a particularly elegant solution that takes advantage of the role swapping of the toroidal meridians and longitudes on inversion. The trick is to construct the two tori separately, inside-out, but *linked*. When two tori are *linked*, the longitudes of one loop around the meridians of the other and vice versa. Whereas when two tori are nested, their meridians and longitudes align, one inside the other. Thus, if we sew the two tori *linked,* as in the configuration in the middle of Figure 7, and then invert one through a small slit, the un-inverted torus winds up nested perfectly inside the inverted one. Seeing this happen is so strange and counterintuitive that there is no substitute for the tactile learning experience of trying it out yourself. We highly recommend this experiment if you have a couple of old cloth napkins and a needle and thread (or sewing machine) on hand. Keep in mind that the

technique is most easily demonstrated using tori whose shape is preserved on inversion. Otherwise, you might wind up trying to nest two very differently proportioned tori, which can be a bit like trying to dress a short fat man in a tall thin man's clothes or vice versa. (For example, consider nesting the two torus shapes in Figure 2b –although, if linked in the shape configurations shown, these two would actually nest nicely because one *is* the inversion of the other!) Using two equal size *square* napkins is one easy way to guarantee that your tori will fit together nicely when nested. Another important detail of the construction is that the two slits should be aligned so that you can access the inner torus slit directly through the outer torus slit, which makes it possible to go back and forth between the two nested states without passing again through the linked state. This works because inverting an interior nested torus forces the simultaneous inversion of the outer torus as well. After sewing and nesting, we typically stitch together the edges of the two aligned slits, so that it is no longer possible to invert just the outer torus and re-enter the linked state in which the fabric backsides and raw seam edges are exposed. After this, without a seam-ripper, re-entrance to the "inner sanctum" is locked off! The graph in Figure 7 shows the possible states of two linked tori, with colored directional links indicating the new state when the torus of that color is inverted.
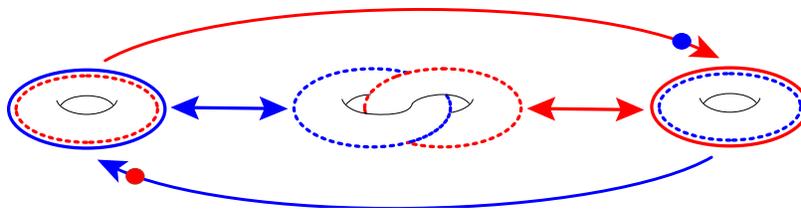
Figure 7: *The nodes in this graph represent the possible states of two linked tori, and the links represent inversion actions. The colored directional links point to the new state when the torus of that link color is inverted. Note that inverting an interior nested torus forces the inversion of the exterior torus as well. Forced inversions are denoted by a dot (the color of the torus being forced to invert) at the base of the link arrowhead. Dashed lines represent torus fabric "back" side out and solid lines "front" side out.*

Appendix A gives instructions using this technique to sew two nested tori with fully hidden seams. To present the idea in its simplest form, the instructions are shown for a scarf made from a square layout with no Moebius twist. However, this same basic technique can also be applied to construct the invertible scarf with hexagonal structure shown in Figure 5. As described in [5], multiple layouts (including a rectangular one) can be used to sew this scarf –and all of them produce a scarf that is equivalent in structure to a hexagon with its opposite edges sewn together (although the seams are in different locations). For these, the linked construction strategy is the same, but other details of the construction are of course different. For example, because the scarf in Figure 5 has a Moebius-like half twist, care must be taken to apply the twist in the same direction for both linked tori (if not, the nested result will seem a bit like trying to put a left-handed glove on your right hand).

How About a Triply Invertible?

Encouraged by our discoveries about the usefulness of inverting linked tori, another challenge was suggested by one of us as a kind of a lark – could we make a toroidal scarf that is *triply* invertible, using three different fabric patterns/layers? Our technique for creating nested tori could easily be applied recursively to produce any number of nested tori, Russian doll style. Might a triple nesting work as a three-way invertible scarf?

One way to convince yourself that it will *not* work is to construct such a triple nesting and try it out. One can also do this as a thought experiment, as outlined in the graph in Figure 8. To create a triple nesting, begin with a nesting of two tori linked to a third torus; then invert the third torus to create the triple nesting. The graph in Figure 8 shows an exhaustive investigation of all possible inversion actions starting this way.

Since it is a complete accounting, the graph demonstrates that there is no sequence of inversions in which the middle torus of a triple nesting (in this case, the green one) is able to make its way to the visible exterior in one of the triply nested states. In other words, only *two* (not three) triply nested states are possible and, in both, the green torus remains hidden in the middle.

Furthermore, additional nestings won't help. Readers may wish to consider how to generalize the proof to nestings of arbitrary size N, but the basic idea is that, for larger N, the graph in Figure 8 stretches to include more linked states in the middle, but the two ends states remain the only ones that are fully nested.
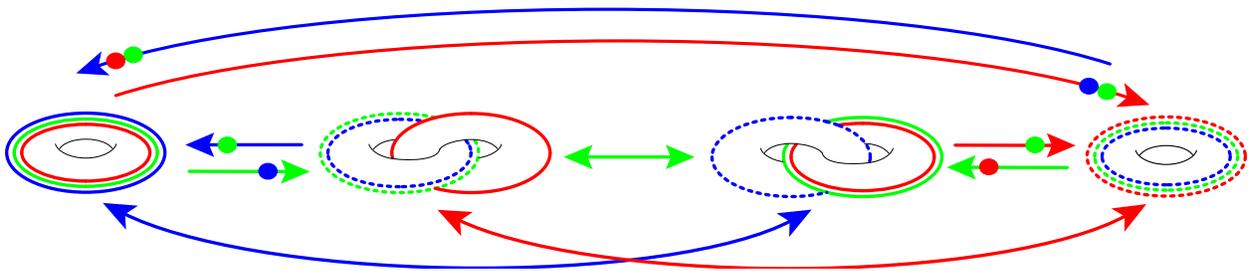
Figure 8: *A graphical proof (that can be confirmed experimentally with sewn fabric) that three fully nested tori can never enter a state (via inversion through a single slit in each) in which the "middle" torus (shown here in green) moves to the visible exterior. The graph shows all possible states and all possible inversion actions from each state. Dashed lines represent a torus whose fabric B side is out and solid lines a torus whose fabric A side is out. Colored directional links point to the new state reached when the torus of that link color is inverted. Note that inverting an interior nested torus forces the inversion of all tori surrounding it in the nesting as well. Forced inversions are denoted by dots (the color of the torus forced into an inversion) at the base of the link arrowhead.*

So Is a Triply Invertible Impossible?

Once again our design aspirations at first seemed stymied, but we went on to consider whether any other sort of linkage involving three tori might work. We began by studying the 3-component links in Robert Scharein's knot zoo (Figure 9 [6]). The links we presumed potentially useful were those with symmetry such that the link relationship between any one torus and the other two is the same regardless of which one you choose. This important symmetry property for a triply invertible scarf guarantees that the nested configuration is identical regardless of which torus is inverted and moved into the exterior position. It was readily apparent that the leftmost three links in Figure 9 all have this property, so those seemed like a good starting place for our investigation. Each of them presents an interesting puzzle in its own right: What happens if you sew three fabric tori linked in the configuration illustrated (Figures 10a-c), and then invert one of them? Figures 10e-g show the solution for each, which can also be confirmed experimentally by sewing and trying it out. The results are shown as X-ray vision drawings to illustrate how the inner tori are positioned after any one torus in the link is inverted.

As it turns out, all three of these linkages can in principle be used to produce a triply invertible scarf, but they are not all equal from a practical design standpoint. To be useful and aesthetically pleasing as a scarf, we also wish to minimize the bulk created by the two inner nested tori, so that the scarf drapes smoothly and is not too lumpy looking. Quantifying this goal more explicitly, we want a nesting configuration in which the fabric of the inner two tori can be distributed as uniformly as possible within the tunnel of the outer torus.
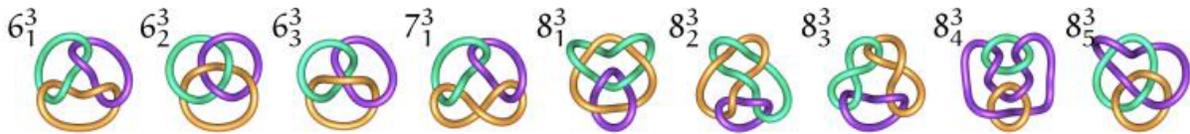
Figure 9: *A section of Robert Scharein's Knot Zoo[6] showing three-ring linkages. Each link in the zoo is labeled X_z^y, where X indicates the crossing number[1], Y the number of components, and Z an archive number among links with that crossing number and component number (Alexander-Briggs notation [2]).*
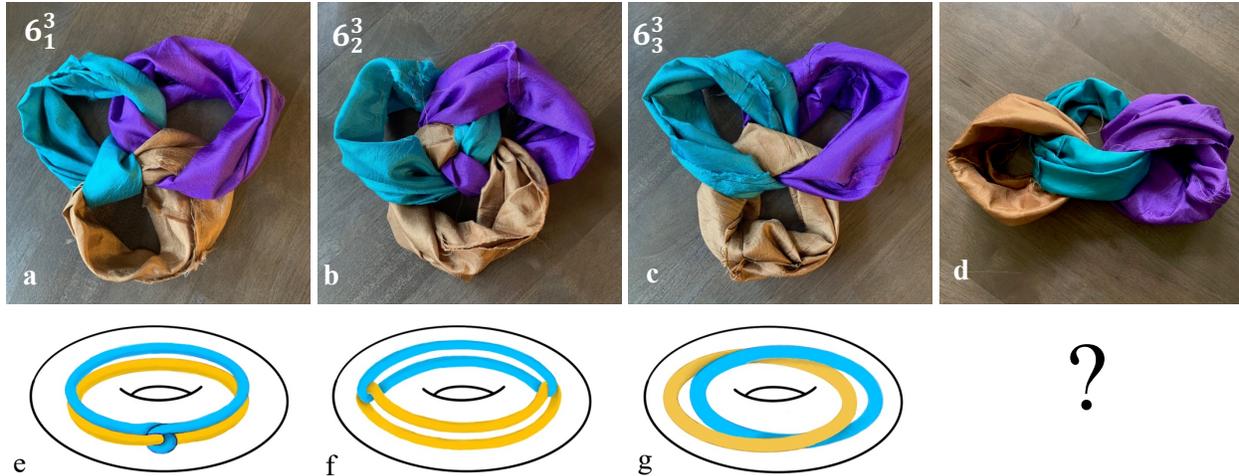
Figure 10: *The 6_1^3, 6_2^3, and 6_3^3 links constructed with cloth tori (a-c) and, below each, an X-ray vision drawing of it after one torus is inverted (not drawn to scale, for illustration purposes) (e-g). A chained link (d) lacks the symmetry required for a triply invertible and thus has more than one possible nested configuration after an inversion. We leave its possible configurations as a puzzle for the reader.*

Ideally the inner bulk would be distributed in the perfectly even layered manner of a Russian-doll-style triple-nesting, as in Figure 8, but we've already shown this configuration doesn't work as a triply invertible (one reason it fails is because it lacks the required symmetry property discussed earlier). Short of that, we want to consider the comparative merits of the linkages in Figures 10e-g and determine which one does the best job of evenly distributing the scarf's inner bulk. It is interesting that the two inner tori in Figures 10e and 10g are topologically linked in the same way, but the outer torus of the 6_1^3 link prevents the blue inner torus from untwisting, making this linkage less evenly distributed than the 6_3^3 link. The interior tori of the 6_2^3 link (Figure 10f) have a different linkage arrangement; in fact, they are not linked at all. (Two rings are not linked if either can be shrunk to a point without passing through the other, as is the case here.) The lack of linkage in this case can be neatly demonstrated by placing two rubber bands in the illustrated configuration. Figure 9 shows that we should expect this result because removing any one loop from the 6_2^3 link frees the other two to separate. When one torus of Figure 10b is inverted, it becomes the outer torus of Figure 10f, which then prevents the inner tori from separating (despite being unlinked) in such a way that their bulk is reasonably well-distributed, making this link a possible contender as a decent triply invertible. Even so, the 6_3^3 link emerges as the clear top contender because its linked interior tori can easily be arranged to distribute in a *perfectly* even fashion along the interior tunnel of the outer torus (Figure 10g). This "perfect" arrangement is most easily seen when the two tori are positioned to touch each other along a (1,1) torus knot [1,4] running along the surface of each (as an example of this positioning, see the two bottom tori in Figure 12d). Thus the 6_3^3 link is an excellent choice for our triply invertible.

More Practical Details

So far, we have only discussed attaching the linked tori together at their slits. On a triply invertible, after one torus is inverted, this creates a double-slotted slit (Figure 11), in which the wearer has a choice about which side of the slit to use for the next inversion. However, other than at the slits, the tori may be left free-floating with respect to one another. A drawback to a free-floating configuration is that the inner tori tend to bunch up during inversions and it can sometimes take a lot of effort to smooth everything out again. We also want to align the slits so the fabric doesn't get torqued when the slit edges are sewn together. The doubly invertible scarf has easy solutions to these problems because its two layers of fabric are always positioned flush back to back and thus can be quilted together at any surface location, or, most readily, along their corresponding seams. Likewise, it is also easy to align their slits (e.g., the method in Appendix A). However, for a triply invertible, which does not use this Russian Doll style of nesting, the layers cannot be quilted together just anywhere, and there is a related issue about where to place the slits. To align its slits, we make use of the configuration illustrated in Figure 12b, where the three tori are positioned so they all touch (or at least come very close to touching) along a (1,1) torus knot on each. Using a parallelogram layout, one can create a seam that falls directly on such a (1,1) torus knot. The Figure 5 invertible scarf has a parallelogram layout (see [5]) in which one seam can be engineered to fall along a (1,1) torus knot, if the Moebius twist runs in the correct direction. The triply invertible scarf in Figure 13 is made this way. Placing the slits along this seam –and then tacking together all three tori along it (while they are "wrong" side out) – solves the slit alignment problem and helps to keep the fabric evenly distributed during inversions. Some shaking out and smoothing by hand after inversions is still needed, however, and we continue to investigate whether there are other better approaches to keeping the inner tori in place during inversions.
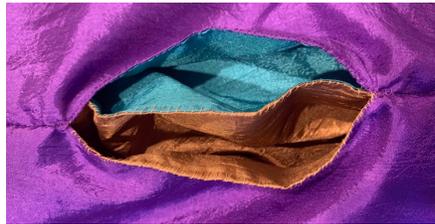
Figure 11: *On a linkage for a triply invertible scarf, with one torus inverted and all three slits aligned, the slit edges may be stitched together. This creates a double-slotted slit through which you can choose which of the two inner tori to invert and expose next.*

Impractical Applications: Scaling Up…to Infinity?

The obvious next question is whether we can scale up. Can we make a quadruple invertible? A quintuple? Is there a limit? In our experience, a triply invertible is at the limit of bulkiness that feels wearable for a scarf – and only works well if made with very thin fabric. But the scalability question is still interesting to consider theoretically, if we imagine infinitely thin fabric. Figure 12 shows a clay model of the same 6_3^3 link in different "poses." The Figure 12c pose suggests how the link can be scaled up while still maintaining the symmetry and inversion properties needed for an invertible scarf. Figure 12d shows how the third ring links through the other two. (Note that this configuration looks the same regardless of which ring we denote as the "third" one because the linkage has our required symmetry property.) We could likewise add a fourth link in the same manner (looped through the other three) by placing it through the hole shown in the center of Figure 12c. And this same operation can be repeated to add a fifth link, and so on. Thus, at least in principle, the 6_3^3 link scales up to an "N-vertible" for any N, while also maintaining both its symmetry property and capacity for even distribution of the (N-1) interior tori.
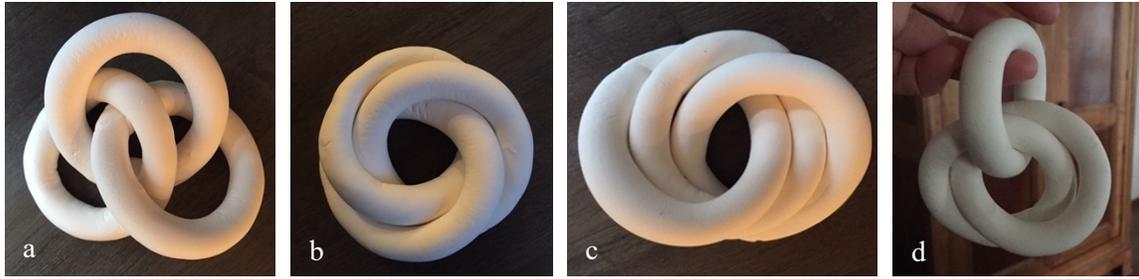
Figure 12: *Multiple arrangements (or "poses") of the same 6_3^3 link using a clay model.*

Conclusions and Future Work

Our design aspirations led to a series of puzzles that we found entertaining and educational as we figured out how to craft the fabric objects we imagined. Despite its possible impracticality fashion-wise, it would be interesting to try out a quadruple or higher number invertible. Perhaps this or other sewing experiments with linked or knotted tori will lead to deeper understanding or reveal more interesting inversion properties of other knots in the "zoo." And while the scarves are great fun, we are seeking ideas for additional possible applications that might exist for multiply invertible objects. We're also curious whether any of these ideas and experimental methods with fabric might be usefully applied to research in knot theory or other areas of mathematics. Finally, we're working on fabric designs that are especially suitable or apropos for triply invertible scarves. Figure 13 shows our first custom designs and a triply invertible scarf made from them that illustrates its own structure –with three fabric patterns depicting three different 6_3^3 link "poses," inspired by the clay model in Figure 12.

Thank you to Kent Christman and Michael Klugerman for their astute observations and helpful conversations.
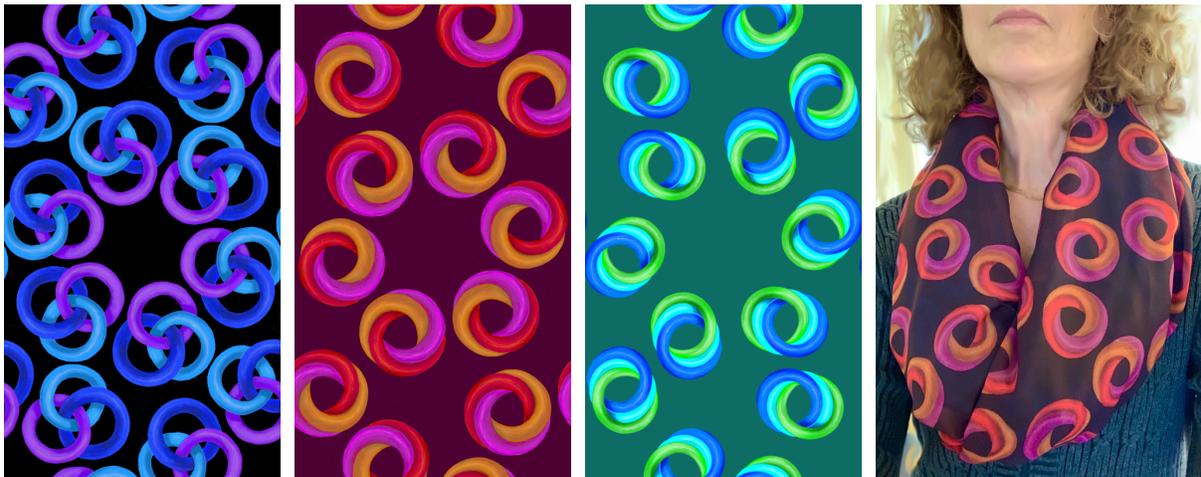
Figure 13: *A triply invertible scarf and fabrics designed for it that illustrate the scarf's linked structure. The design motifs were hand-drawn and organized as a repeating pattern using the app iOrnament.*

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[7] Surot. Public domain animated gif at: https://en.wikipedia.org/wiki/Torus#/media/File:Inside-out_torus_(animated,_small).gif (as of January 14, 2020)

1. Sew together the opposite edges of two square pieces of fabric to form two "cylinders," leaving an open slit midway on both seams.

2. Invert one of the cylinders so that it is right-side out.

3. Cuff it over to bring the top edge of the cylinder toward the bottom edge.

4. Align the top and bottom edges and stitch them together all the way around to create the first (inside out) torus.

5. Place the remaining cylinder through the hole of the torus you have just made.

6. Bring the cylinder top and bottom edges around in an arc to meet each other…

7. …with the right-sides of the fabric touching to form a flange. Stitch in place all the way around the flange to create a pair of linked inside-out tori.

8. Invert one torus through its slit to nest them. Align the two slits and stitch the slit edges together.

Appendix A*: Instructions for sewing an invertible torus with hidden seams made from two square pieces of fabric. For illustration purposes to show the seam joint more clearly, the gold torus in the two bottom center figures is obviously not drawn to scale (it's stretched longer). Note that using these instructions, in the linked configuration, the slit on one torus winds up on a longitudinal seam while the slit on the other torus winds up on a meridian seam –this aligns the slits in the nested configuration.*