

Exploring Symmetry Using Aestheometry in Classrooms and Beyond

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Abstract

This workshop shows how to use the *4ColorAestheometry* Excel file to teach students various geometric notions involving symmetry in a smart classroom. The file allows open exploration of a variety of geometric and algebraic topics and may be used from the late primary through secondary levels of the mathematics curriculum.

1. Introduction and Motivation

Aestheometry is the art of creating patterns and curved images using straight lines. One popular method of doing aestheometry is based on string. Curve stitching, developed by Mary Boole more than a hundred years ago [1, 6], has been elaborated on in various ways, including string art popularized in the 1970s [11] and, more recently, with pen and paper [5, 9] and with computer programs [3, 10]. This concept has been termed aestheometry, curve stitching, rhythmic geometry, parabolic art and string art.

Traditional string art connecting non-parallel lines produces a parabolic arc like in Figure 1(a) or around a circle creates a cardioid like in Figure 1(d) [4, 8, 10]. However, individuals need not have prior knowledge of geometric concepts like parabolas or cardioids to have fun with string art. The *Aestheometry* Excel file was developed to allow students who have just learned about Cartesian coordinates the ability to independently explore graphing and geometric and numeric pattern recognition outside the classroom, yet it is easily adapted to an Excel enabled classroom setting [2]. The file mimics string art on a closed set of vertices, like the octagons in Figures 1(b) and 1(c). The *4ColorAestheometry* file extends the original file in order to focus on symmetry. Excel was chosen due to its ubiquity, user-friendliness, and scalability.

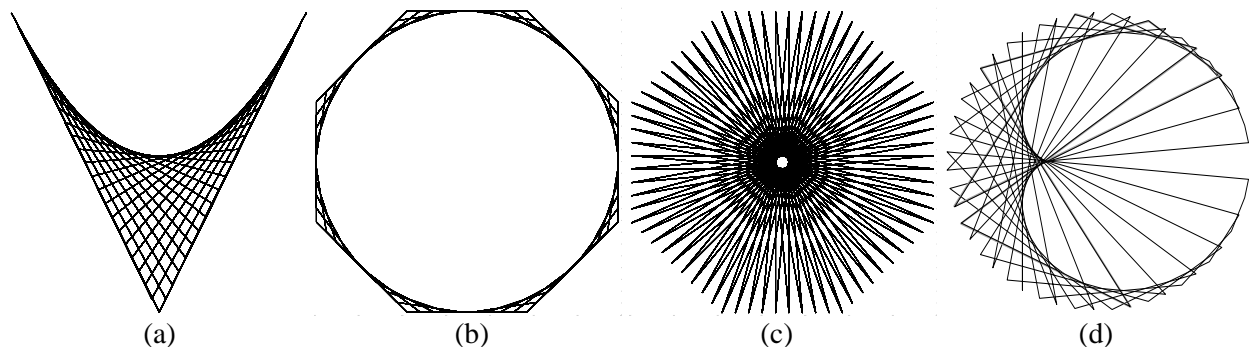


Figure 1: Some basic examples of Aestheometry.

Students of all math levels and Excel-proficiencies are able to learn and engage with this file due to the simplicity of three controllable parameters: *Vertices*, V ; *Subdivisions*, S ; and *Points*, P . (The Excel file also includes two 1-page handouts, one for basic exploration and the other for more advanced discussion.) In this workshop, beginning in Section 2, we define these parameters and observe that changing any of the three alters the entire image, even when the underlying vertices do not change. While holding two parameters fixed, we focus on how the third alters the image, such as how the “circle” in Figure 1(b) turned into the star in 1(c) by changing $P = 5$ to $P = 35$ (given $V = 8$ and $S = 9$). In Section 3, we examine how the

image changes when underlying vertices change, holding all three parameters fixed. Section 4 extends the basic model by examining four-color extensions of the image, and Section 5 shows the user how to purposefully create asymmetry in an image. In each section, we will focus attention on the various forms of symmetry that are inherent in the emerging patterns. The figures below suggest values for each workshop exercise; however, the file allows for open-ended exploration either in a computer lab or in a workshop where individuals use their own Excel-enabled device. The goal of this workshop is to guide participants in much the same way that teachers guide their students through new material. For instance, the workshop presenters have taught classes using this file and have helped students understand why patterns may radically change when a parameter increments by one (for example, the star in Figure 1(c), produced from $P = 35$, transforms into a single line when $P = 36$, since 36 is half of $V \cdot S = 72$ in that figure).

2. Exploring the Interrelation between Vertices, V, Subdivisions, S, and Points, P

Defining V, S, and P

To explain the difference between Vertices, V, and Points, P, one must first understand Subdivisions, S. Formally, S is the number of (equal) subdivisions between two consecutive vertices. Figure 2 is an image from Excel sheet *3a.Square* which annotates V, S, and P. In that figure, $S = 2$, and we see two equal-sized subdivisions, **S.1** and **S.2**, on each side of the square. The square has four vertices, $V = 4$, laid out in clockwise fashion starting at **V.1** = (0, 0) in the lower-left corner (the coordinate system is suppressed to reduce clutter in all figures). Vertices provide the skeletal frame on which the image is created. For easy reference, color-coded definitions of the three parameters are provided to the right of the image in Figure 2.

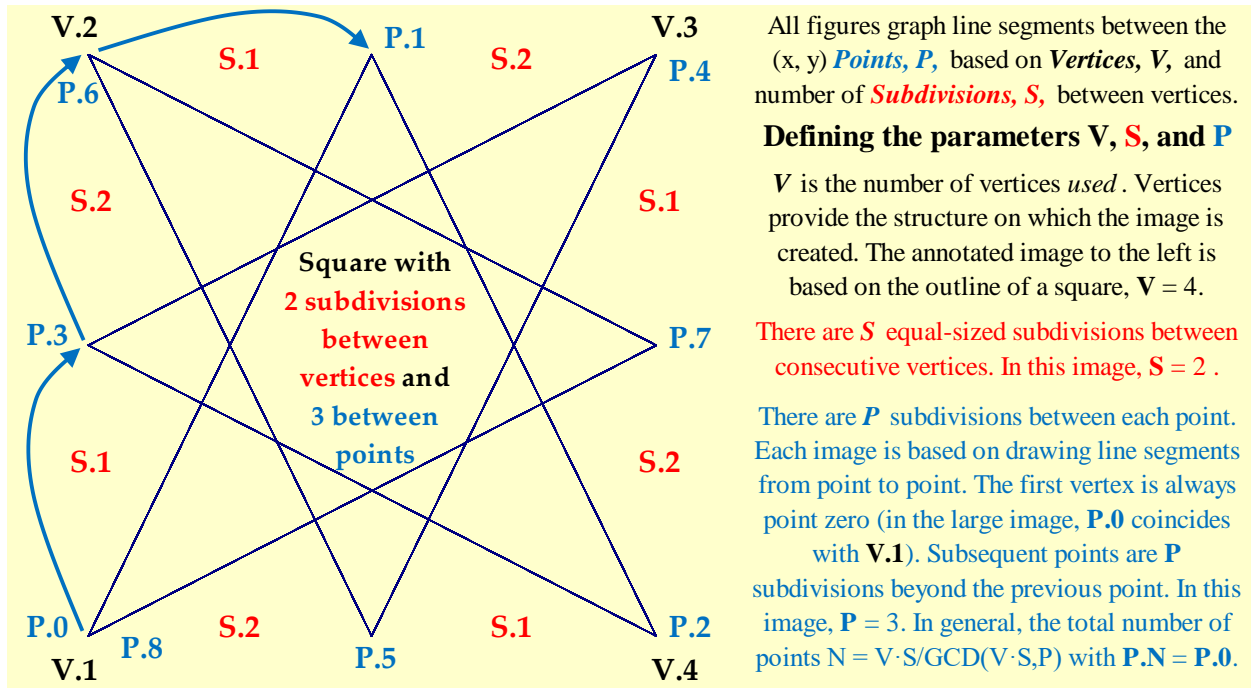


Figure 2: Annotated image based on unit square with $V = 4$, $S = 2$, and $P = 3$.

Each image in this workshop is based on drawing line segments from point to point. The first line segment, from point zero to point one, is determined in the following fashion: point zero is always the first vertex, **P.0** = **V.1**, and point one, **P.1**, is P subdivisions past **P.0**. In Figure 2, $P = 3$. Point two (and second line segment) is three subdivisions past the first (at **P.2** = **V.4**) and **P.3** is three past **P.2** and so on until the **P.8** coincides with the initial vertex (**P.8** = **V.1** = **P.0**). In the end, an eight-pointed star is mapped out by circling the square 3 times. The resulting image is the eight-point Brunes Star [7]. This image is both horizontally

and vertically symmetric. Each vertex and each subdivision point are used because there are no common divisors between P and $V \cdot S$. The number of distinct line segments, $N = V \cdot S / \text{GCD}(V \cdot S, P)$, where GCD is the greatest common divisor function. If any of the three parameters V , S , or P change, the image will change. The next three activities vary one parameter at a time, holding the other two fixed.

Exploring the Number of Subdivisions between Points, P

When the number of subdivisions between points, P , is less than the number of subdivisions between vertices, S , the resulting image will always contain the outline of the line segments connecting the vertices as part of the image. Figure 3(a) provides an example because $P = 11 < 19 = S$ (using the *3a.Square* sheet of the Excel file). Note that the frame of Figure 3(a) is smooth, rather than ragged like in Figures 2, 3(b), and 3(c) where $P > S$. The curves in each corner of Figure 3(a) create a shape that “roughly” resembles a circle, but, in fact, is the amalgamation of four parabolas (and 1(b) is an amalgam of eight parabolas). The parabolas do not appear to overlap, however, that is not quite true since P is more than half of S . Each curve in Figure 3(a) is created from 10 line segments that are not part of the horizontal and vertical frame.

When $S < \text{MOD}(P, V \cdot S) < (V-1) \cdot S$, no line segment lies on the frame created by the vertices (where MOD is the remainder function). This is because each subsequent point MUST lie on a different part of the frame. (The reason for the restriction on P is easy to see by simply checking $P = 65$ and $P = 87$ in cell C1. In both instances, the resulting image appears identical to Figure 3(a) because $65 = \text{MOD}(65, 4 \cdot 19)$ and $11 = \text{MOD}(87, 4 \cdot 19)$.) Such images might be considered “star-like” because of the pointy subdivision points. Clearly, Figure 3(c) is much pointier than 3(b) yet both could be described as 76-point stars.

It is worth pointing out a couple of other things about these images. There is an internal square in Figure 3(b), one subdivision on each side away from the nearest vertex. Note, for example the horizontal line going from $(0, 18/19)$ to $(1, 18/19)$, just below the second and third vertices. This segment was created with 1 subdivision on the left edge, 19 subdivisions on the top, and 1 subdivision along the right edge. If you try $P = 23$, you will find a similar internal square two subdivisions inside of each vertex. The intersecting curves in Figure 3(b) are each created from 17-line segments because the first segment begins two subdivisions past the second vertex. Figure 3(c) is the star with sharpest points available given $S = 19$, since $37 = V \cdot S / 2 - 1$. When $P = 38$, the image is a single line ($38 = 4 \cdot 19 / 2$) and when $P = 39$, the image is identical to $V = 37$, but now 39 revolutions around the square are required to create the image instead of 37 revolutions around the square.

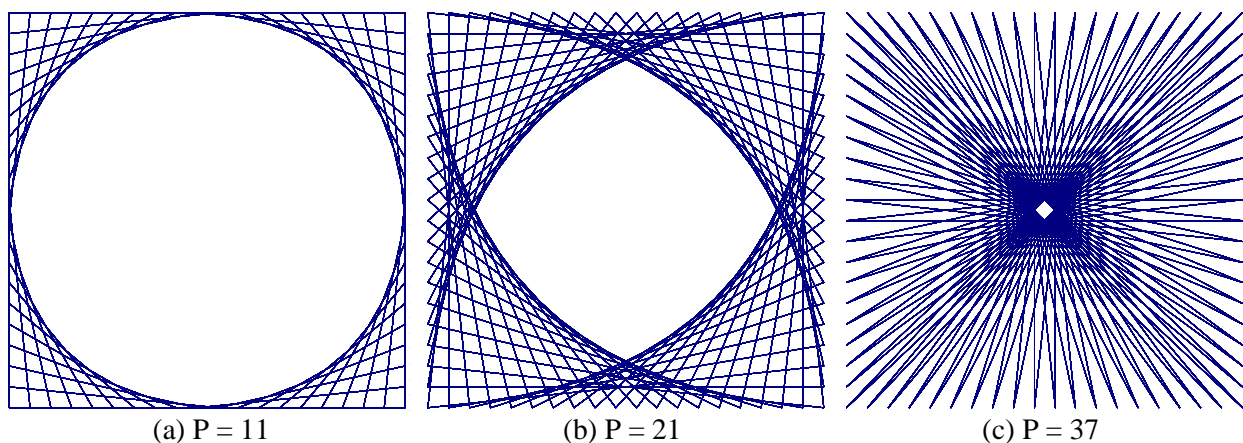


Figure 3: Images based on square vertices with $V = 4$ and $S = 19$.

Exploring the Number of Vertices Used, V

Varying the number of vertices used, V , shows how the hexagonal image in Figure 4(d) was built using the *5c.6Petal* sheet of the Excel file. (To see why this is called 6Petal, set $P = 11$, the same S and P values shown in Figure 3(a).) Vertices are at even hours on the clock-face. The first four vertices are at 2, 8, 10, and 4 o'clock: the “bow tie” in 4(a) emerges. The next two vertices alternate adding 2 hours and 6 hours,

so 4(b) has additional vertices at 6 and 12 o'clock. The remaining six vertices require us to “reverse course” by alternatively subtracting 2 hour and 6 hours, thus 4(c) added vertices at 10 and 4 o'clock and 4(d) added vertices at 2, 8, 6, and 12 o'clock. In each instance, the last vertex used connects back to the initial vertex (at 2 o'clock) in order to create a closed set of vertices. The four seemingly parallel crossing sets of lines in 4(a) occur because $P = 23$ exceeds $S = 19$ by 4 in each panel.

Consider three forms of symmetric reflection: horizontal, vertical, and point. Figures 4(a) and 4(d) exhibit all three reflective forms. Figure 4(b) exhibits horizontal but not vertical or point reflection, and Figure 4(c) exhibits point but not horizontal or vertical reflection. Of course, if we broaden our notion of reflective symmetry to include axes that have been rotated, then Figure 4(c) has 60° rotated symmetry.

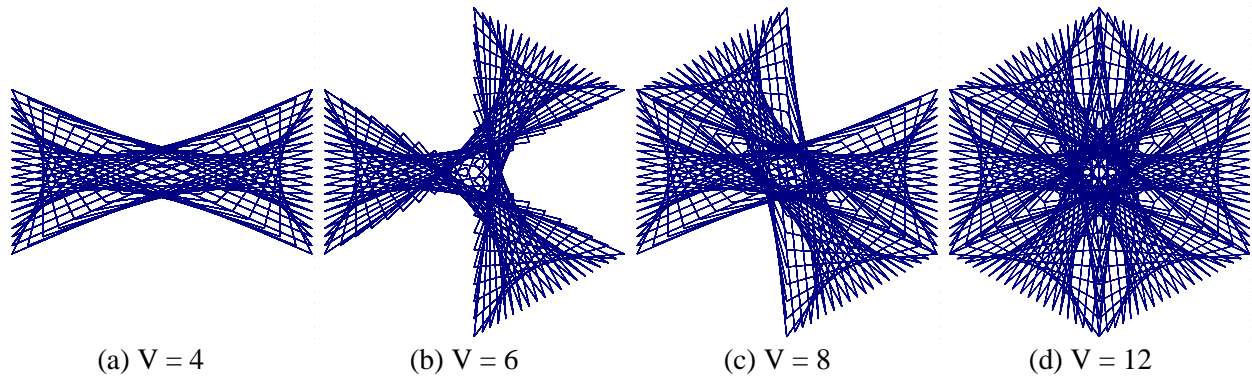


Figure 4: Pattern emergence based on underlying hexagonal petal vertices with $S = 19$, and $P = 23$.

Exploring the Number of Subdivisions between Vertices, S

For fixed V , the number of subdivisions between vertices, S , determines the density of segments in the image since the number of distinct segments is inversely related to $\text{GCD}(V \cdot S, P)$. This is examined using the *5d.8Petal* sheet in the Excel file. In Figure 5(a), $S = 8$ can be seen in two ways: 1) each side of the octagon has 8 readily visible subdivisions, and 2) there are four distinct sized octagonal patterns in the image, the first being the outer edge and the fourth being the almost octagonal pie-plate in the center (a close examination shows that this is, in fact, 16-sided because each outer “edge” is two crossed lines). When $S = 12$, the result is a very sharp 96-point star; Figure 5(b) is extremely similar to Figures 1(c) and 3(c). The difference, other than the overall shape in 3(c), is that there are four darker lines going from vertex to opposing vertex. These lines are darker because there are subdivision points along the four diagonal lines connecting vertices, such as the one from about 1 o'clock to 7 o'clock. The final image, Figure 5(c), has a hollow center because S is odd.

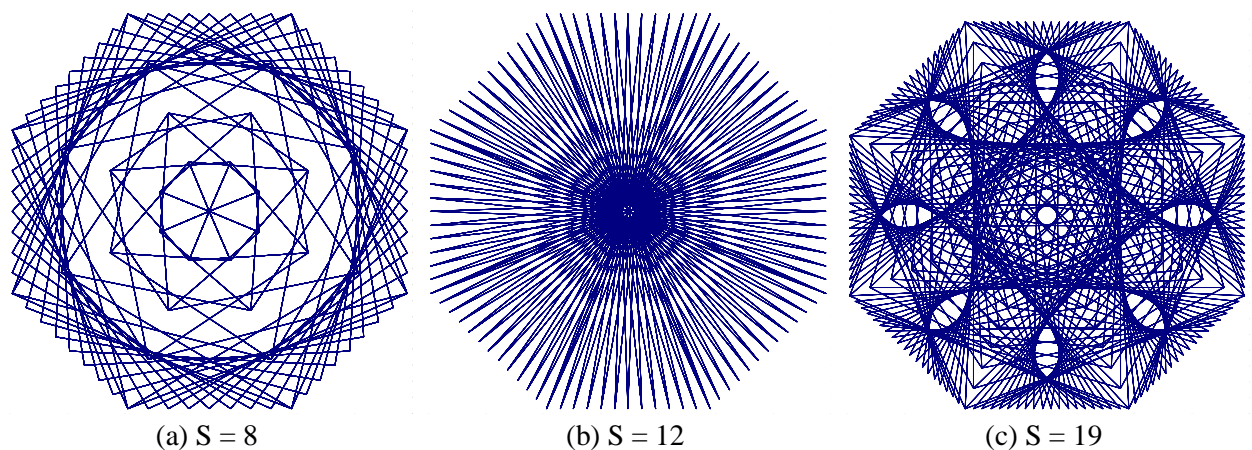


Figure 5: Images based on octagonal petal vertices with $V = 16$ and $P = 97$.

3. Exploring Linked Vertices

We have seen how V , S , and P change a graphic image, but what happens if we change the location of the vertices themselves? Quite clearly, the images will change. That change can provide further opportunity to explore the richness of the *4ColorAestheometry* file. The petals in Figures 4 and 5 are equal in size, meaning each side of the hexagon and octagon are the same size, regardless of whether that side is horizontal, vertical, or diagonal. But this need not be the case, as can be seen in Figure 6. The four images in Figure 6 differ only in the location of the initial y value. Figure 6(c) uses the same initial vertices as Figure 5, which is why the petals are equal sized. The vertex is at $(0.9, 0.9/(1+2^{0.5})) = (0.9, 0.3728)$.

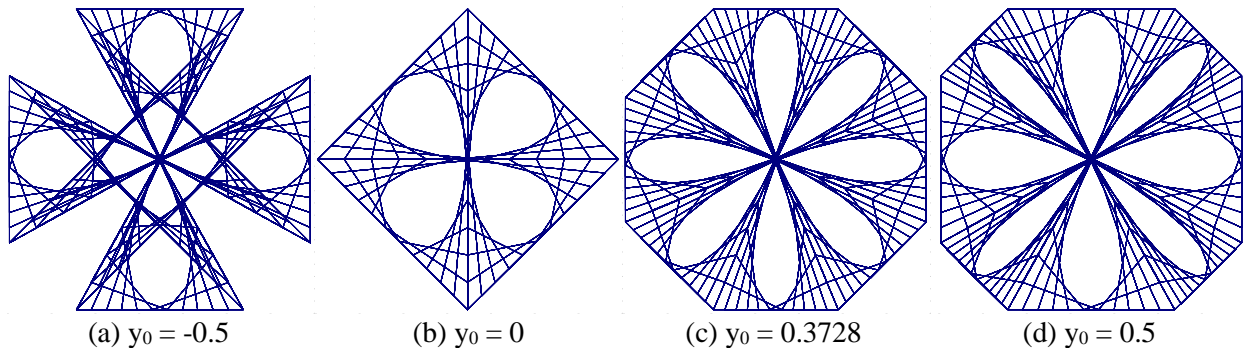


Figure 6: Linked vertices at $(\pm x_0, \pm y_0)$ and $(\pm y_0, \pm x_0)$ given $x_0 = 0.9$, $V = 16$, $S = 11$, and $P = 7$.

Figure 6 was created using the *5d.8Petal* sheet in the Excel file. By changing a single number, the initial y value in cell W4, we can alter the entire image since the initial x and y values are linked to locations in the vertices list in an appropriate order. (This is akin to how the hexagonal image in Figure 4 was created.) You can see the vertex order if you type $V = 4, 6, 8, 10,$ and 12 in cell A1 and examine the resulting images. In each instance, you should be able to note the kind of symmetric reflection inherent in the image: horizontal, vertical, point, or rotational. ($V = 14$ does not work because $\text{GCD}(V, P) = 7$ when $V = 14$.)

In contrast with the equal-sized petals in Figure 6(c), Figure 6(d) with $y_0 = 0.5$ has smaller diagonal petals and larger horizontal and vertical petals since the distance between horizontal and vertical vertex pairs is 1, however, the distance between diagonal vertex pairs is $0.566 = 0.4 \cdot 2^{0.5}$. The Iron Cross in Figure 6(a) has the same sized horizontal and vertical petals as in Figure 6(d), but the diagonal petals are more difficult to see since the initial y placed the initial vertex in the 4th quadrant rather than in the 1st. The diagonal petals are visible, but they are smaller in height and span horizontal and vertical petals (for example, the diagonal petal at about 1 o'clock to 2 o'clock in Figure 6(d) spans from about 11 o'clock to 4 o'clock in Figure 6(a)). There are NO horizontal and vertical petals in Figure 6(b) because $y_0 = 0$. You could just as easily have eliminated the diagonal petals by setting initial $x_0 = \text{initial } y_0$.

4. Using the Four-Color Open Vertices Model

The main sheet, *6.FourColor*, allows you to create independent images with vertices of your choosing. This provides great flexibility to create images with or without symmetric content. Instead of one value of V , S , and P , there are four, one for each color in cells A1:C4. And, instead of a maximum of 21 vertices and 1,000 segments, you can create images with up to 40 vertices and up to 2,000 segments for each color.

The sheet opens with vertices that show the user how to create tessellations of a single image; in this case, red is the parent image. This setup allows you to create a different kind of 8 petal flower from Figures 5 and 6: one with alternating shaped petals. For example, the diagonal petals in Figure 7(b) are heart-shaped because there are fixed diagonal vertices at $(\pm 1, \pm 1)$ and at $(0, 0)$ in the red image. By contrast, the horizontal and vertical petals are elongated footballs much like Figure 3(b) since P is slightly larger than S .

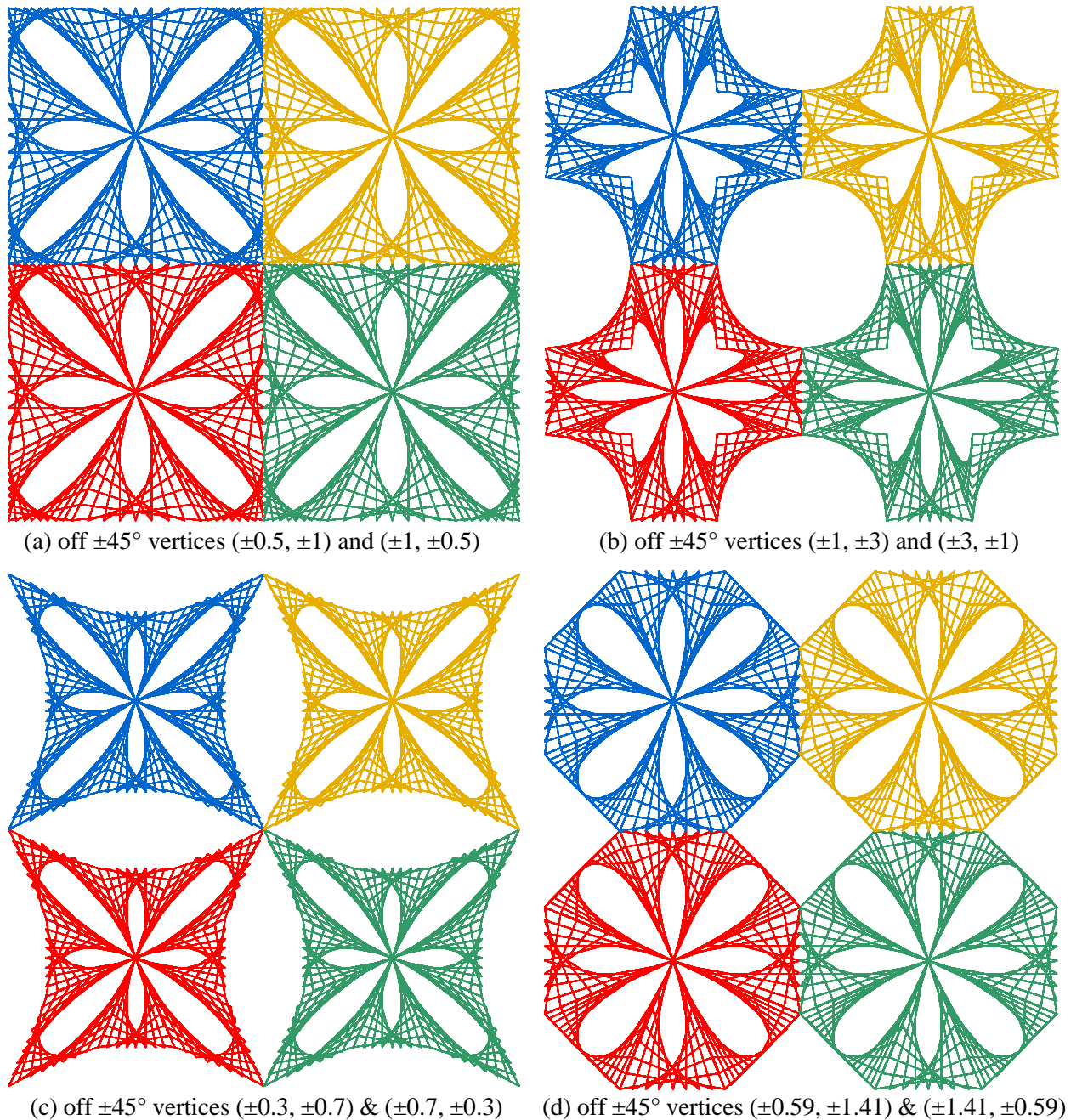


Figure 7: Two four-color images showing symmetry within and across color based on the same underlying vertices. In all panels, red is the “parent” image and $V = 28$, $S = 8$, and $P = 9$.

The center of red’s flower is set to $(0, 0)$ and off $\pm 45^\circ$ vertices are based on the values in B50:B51 which are linked to the red vertices in B9:B36. The other three images are shifted over, or up, or over and up by a shift factor, which is shown in E6:L6. To have the other images touch one another, the shift factor must be twice the maximum of 1, B50, and B51 ($=2*\text{MAX}(1, \text{B}50, \text{B}51)$ is in cell E6). The shift factors are added to the red x and/or y values to create the new color’s values. Note that this addition uses an equation you may have to explain to students if they do not know about Excel’s use of \$ signs. The x coordinate of green’s first vertex in cell E9 is given by the equation $=\$E\$6+\text{B}9$. The value in cell E6 is the x shift factor. This must be added to the ENTIRE set of red x values to create green x values. You can drag the equation in E9

down to E36 and avoid having to type the equation 28 times. Excel uses \$ signs in equations to fix the referenced cell upon dragging. This is called *Absolute* cell referencing, and, as this equation is dragged down, B9 will change to B10, B10 will change to B11, etc. but E6 will not change because of the \$ signs in front of the E and the 6.

When $B50 < 1$ and $B51 = 1$, the edges are in contact with each other and a square pattern emerges as seen in Figure 7(a). The “rope” border around the “circular” center in Figure 7(b) is created from four right-angle corners with P one larger than S (9 versus 8). The vertex corner is (1, 1), (1, 5), (5, 1), and (5, 5) in the red, blue, green, and gold image, respectively. Figure 7(c) only touches at corners because both the x and y off $\pm 45^\circ$ vertices are smaller than 1. The outer vertices of Figure 7(d) are based on $x = 2^{0.5} = 1.41$ and $y = 2 \cdot 2^{0.5} = 0.59$ in order to be in line with the (1, 1) fixed vertex. The result is a regular octagon with sides $4 \cdot 2 \cdot 2^{0.5}$ long. Thus, it has a similar structure to Figures 5 and 6(c), but, as noted above, the petal structure alternates here. The exterior of the octagon has 12 vertex points because $(\pm 1, \pm 1)$ is the midpoint of red’s diagonal vertices. By contrast, horizontal and vertical petals have no midpoint vertices.

Vertices need not form regular polygons; indeed, they need not appear like polygons at all. The parent red skyline is in the first quadrant in Figure 8. As a result, it is simply a matter of changing (additive) shift factors in E6:L6 into (multiplicative) scaling factors. The green horizontal reflection was accomplished by multiplying red’s y values by -0.6 in Figure 8(a). The gold vertical reflection used a scaling factor of -0.5 on red’s x values and the blue image used both scaling factors. In Figure 8(b), x and y are switched in the 2nd and 4th quadrants, $(y_{\text{red}}, -x_{\text{red}})$ produces green and $(-y_{\text{red}}, x_{\text{red}})$ produces gold. The result is point symmetric.

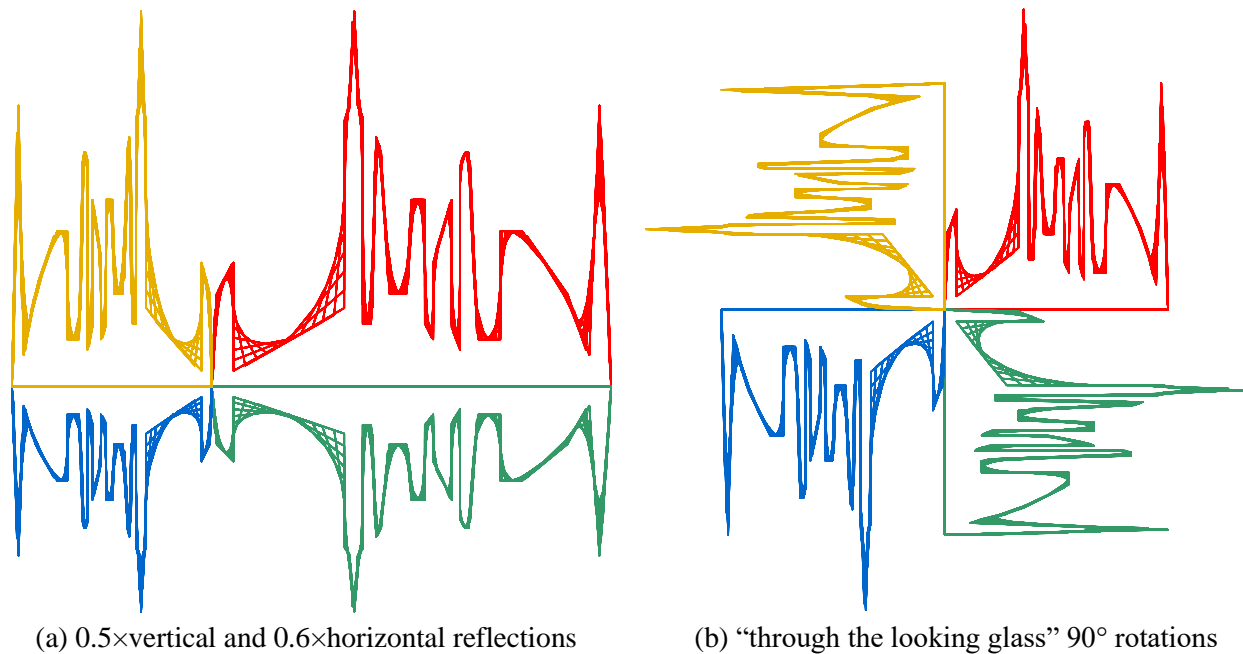


Figure 8: Two four-color cityscapes with $V = 40$, $S = 10$, $P = 7$.

5. A Trick that Introduces Asymmetry

If you list the same vertex twice in a row, you capture the curve at that vertex as can be seen in Figure 9(a). Figures 9(a)-9(c) have the SAME values of S and P as Figures 3(a)-3(c). Figure 9(b) has a seemingly erratic single line running from one subdivision before to one subdivision after the point (1, 1) but this is because 1 subdivision *before* + 19 subdivisions *at* + 1 subdivision *after* = 21, as demanded by $P = 21$. If you check $P = 23$ you will find 3 lines beginning to create the emergent curve at (1, 1). Figure 9(c) shows a curve that is 18/19 of the top and right side (because $18+19 = 37$), and the image near the origin is like Figure 3(c). Each image is symmetric on the 45° line but not horizontally, vertically, or reflected through a point.

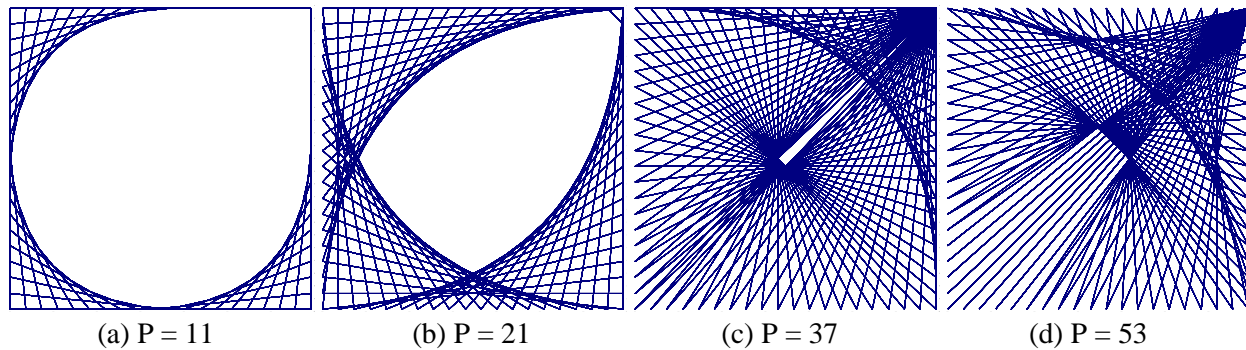


Figure 9: Asymmetric square created by two consecutive vertices at $(1, 1)$ given $V = 5$, and $S = 19$.

Summary and Conclusions

This workshop allows individuals the opportunity to explore notions of symmetry while learning about the *4ColorAestheometry* Excel file, which was used to create all images in the paper (except Figure 1(d)). This file is an extension of the *Aestheometry* file that was originally created to allow students to independently explore topics in math and simultaneously create art [2]. These files can be used to springboard classroom discussions about graphing, symmetry, and pattern recognition as well. They can be used at a variety of grade levels since many of the worksheets are controlled by three numbers. Likewise, many patterns can be explored without formally having been taught about Cartesian coordinates. Our workshop offers a user experience which can be tailored to the interests and skill-level of any user. In contrast with other methods of integrating aestheometry into the classroom, students in this workshop spend the majority of their time creating different patterns and comparing them, rather than creating one object and observing it [8].

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