Bringing Orbifolds out of the Plane: Kaleidoscopes, Gyrations, Wonders, and Miracles

J. Ben Gould[1] and S. Louise Gould[2]

[1]Pathways in Technology Early College High School, Brooklyn, NY; bgould@ptechnyc.org
[2]Professor Emerita Department of Mathematical Sciences, Central Connecticut State University, New Britain, CT; goulds@ccsu.edu

Abstract

Mathematics teachers, artists using mathematical symmetry in their work, and curious mathematicians can benefit from becoming familiar with John Conway's unifying work on symmetry. Conway expanded on William Thurston's concept of orbifolds by developing a set of notations, along with an accounting system for them called "cost," which allows simple arithmetic to be used to determine what types of patterns are possible. Conway and Thurston described many types of spaces, but we will focus only on patterns in the plane. Conway presents orbifolds as objects made by repeatedly folding patterns onto themselves until there is just one copy of a fundamental region. This workshop will provide hands-on experience with the orbifold signatures and calculating the cost of a pattern. We will introduce tools for designing patterns and build physical orbifolds from specific patterns.

Historical Note

One of John Conway's great gifts was his ability to make difficult mathematics more accessible to more people. When we developed this workshop in February 2020, a principal goal was to do our part to pass on Conway's work by helping participants to experience his genius in a physical, hands-on way. With news of his passing this April, it seems even more relevant. He will be greatly missed, but his influence will live on at Bridges and beyond.

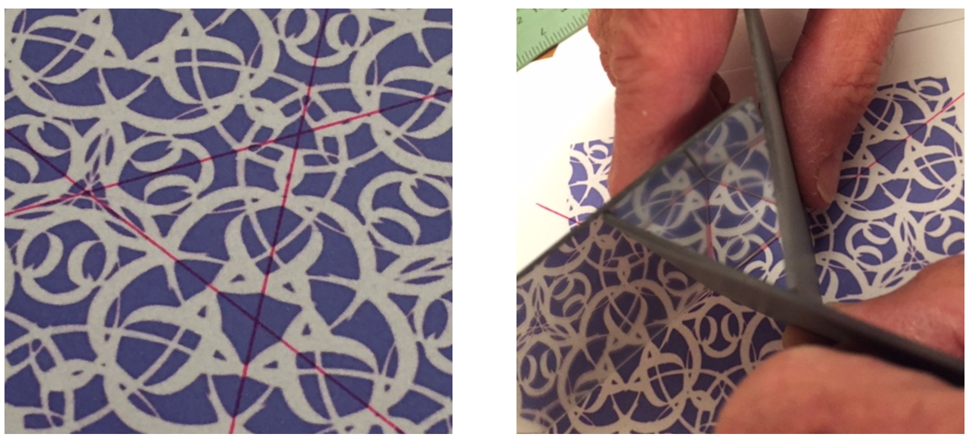
Figure 1: *Examples of folded orbifolds.*

Workshop Overview

This workshop applies much of the vocabulary and ideas presented in Conway's *The Symmetry of Things* [2] related to wallpaper patterns (i.e. those in the plane). Our goal is to deepen participants' understanding of wallpaper patterns, their orbifold classification, and their mathematical structure through the use of hands-on activities. The activities can be used or adapted by classroom teachers, artists, and others to build their understanding and skills with the orbifold notation. What is new in this presentation is the actual paper construction of orbifolds from the identification of fundamental regions and their connections to the patterns in the plane.

Participants should walk away with:

- The ability to determine and write the orbifold signature of any wallpaper pattern
- An intuitive understanding of orbifolds
- Some folded-up orbifolds
- An informal argument that there are exactly 17 wallpaper groups
- Handout materials to use in classrooms or studios. Full-sized handouts are available are available in the supplementary file.



(a) (b)

Figure 2: *(a) Mirror lines drawn on a wallpaper pattern, and (b) acrylic mirrors on mirror lines. The pattern shown here has signature *333.*

Detailed Sequence of Activities

Exploring Mirror Lines

Participants will work in pairs to explore mirror lines on paper and with physical acrylic mirrors.

1. Distribute several large-sized versions of patterns from Figure 3, that include mirror lines that are not all parallel.
2. Use colored pencils and rulers to draw mirrors on the patterns. It is not necessary to draw every mirror line, but draw at least one of each non-parallel line, and the region enclosed by the intersecting lines should not contain any other mirror lines.
3. Place the acrylic mirrors along the mirror lines we have drawn. By leaning from side to side, verify that the pattern appears in its entirety when looking into the space formed by the mirrors. In fact, if the mirrors were perfect, the pattern would extend infinitely. Figure 2b shows an example of this.
4. Discuss angles formed by the intersecting mirror lines and the implications for physical kaleidoscopes. Mirror lines can intersect at 90 degrees, 60 degrees, 45 degrees or 30 degrees. There are no other possibilities. Discuss why this is so.

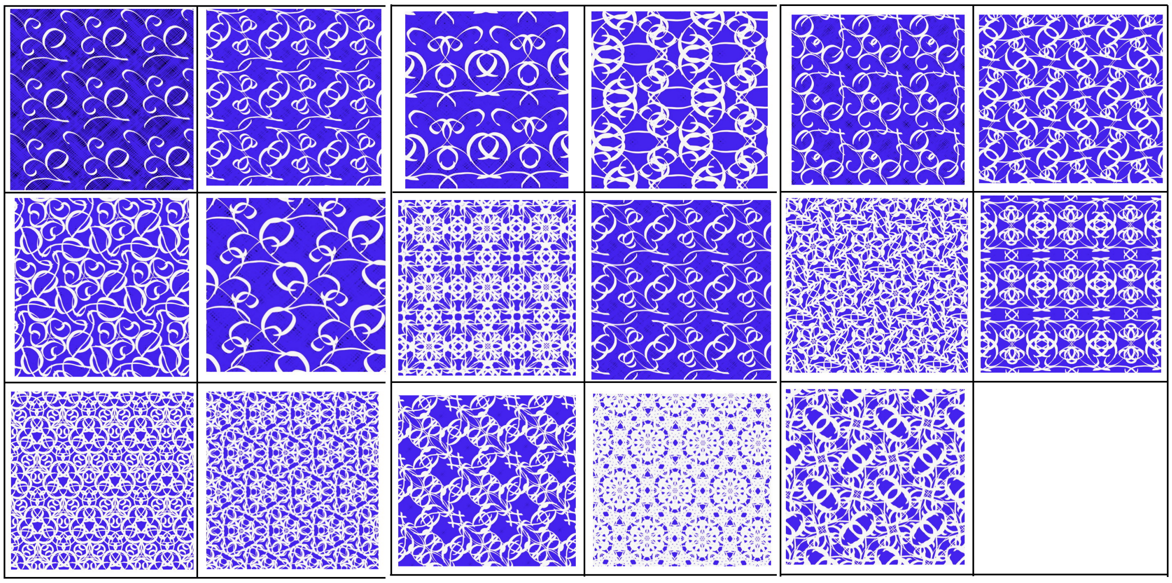


Figure 3: *Seventeen variations on a theme. The patterns were developed using iOrnament [5].*

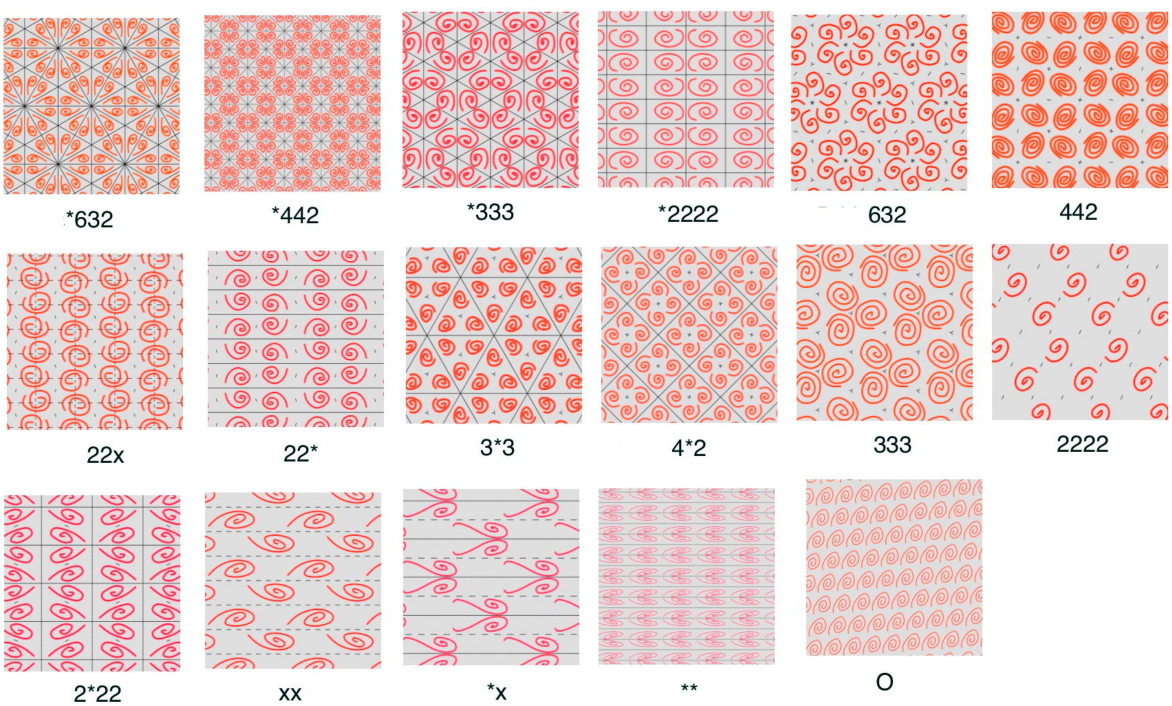


Figure 4: *Patterns created with KaleidoPaint [8] that represent each of the 17 wallpaper groups, labeled with their orbifold signatures. They also show centers of rotation, mirrors and glide reflections.*

Finding and Marking Features of a Wallpaper Pattern

1. Discuss labeling mirror lines, and the vertices of intersecting mirror lines (kaleidoscope points). The vertices should be labeled with a 2, 3, 4, or 6, depending on the rotational symmetry order.
2. Distribute copies of a pattern with the *× signature, and discuss how to label mirror lines without intersections, along with how, in this case, a physical kaleidoscope would be a pair of two, infinitely long mirrors.

3. Discuss “miracles” and how to find them. Label the miracle on the *× pattern. These are glide reflections, which Conway calls miracles [2, p. 31] because a flipped over copy of a point occurs without reflecting in a mirror line. Look for a motif that curls clockwise and a nearby copy of the same motif that curls counterclockwise. When the two points are not on opposite sides of a mirror line, this represents the existence of a miracle.
4. Distribute copies of a pattern with the 22× signature. Label the miracle. Discuss and demonstrate labeling gyration points, points where there is rotational symmetry but no mirror lines.
5. Finally, discuss patterns that have no mirror lines nor miracles, and indicate how you would indicate a “wandering,” that is a repetition not explained by mirrors, gyrations, nor miracles. This is what you have if you can travel to a copy of a point (not flipped) in two different directions without crossing a mirror line, and without the presence of any flipped-over versions of the point.

Card Matching Game

1. Pairs of participants are provided two sets of cards: a set of unlabeled blue card (Figure 3), and a set of red cards, already labeled with their orbifold signatures (Figure 4).
2. Participants can mark up the blue cards in order to determine their signatures.
3. The goal is to match up each blue card with the corresponding red card that has the same signature.

Making Physical Orbifolds

For all of these, use large copies of the patterns available in the handouts. Because we cannot actually fold up an infinite pattern, the key is to cut out enough of the pattern that we can see how the entire pattern could be folded into its orbifold, while still being able to do it with paper. Detailed instructions for making some of the orbifolds are provided below. Symbols in the orbifold signature translate into features of the orbifolds. Mirror lines become edges, gyrations become cone points, kaleidoscope points become corner points, and miracles indicate the topology has been changed to a non-orientable surface: a Möbius band, Klein bottle, or projective plane. The angles around cone points and at corners are determined by the number associated with the gyration or kaleidoscope point. For example, the orbifold for 4*2 will have one 90-degree cone point, one edge, and one 90-degree corner point (see top right of Figure 1).

*333: Make sure mirror lines are drawn (Figure 2a). Cut along mirror lines, leaving at least three copies of the pattern. The remaining copies are folded on mirror lines until there is a single triangle. Fold the paper so that the pattern is visible on both sides. The two sides are identical mirror images.

333: Mark and circle one gyration point. Find and mark the 6 other nearest gyration points that form a hexagon around the circled point. Add one more gyration point outside this hexagon and cut out the polygon defined by these points (a hexagon with a triangle attached). Cut a slit from one of the outer gyration points of the hexagon to the center. Overlap the two sides of the slit and form a tighter and tighter cone until only one copy of the pattern is visible around the point. At this point, there should be a cone with the extra triangle hanging off of it. Flatten the cone into an equilateral triangle, and the extra triangle should fold down and match up with the part of the pattern that is already there, demonstrating the other points are, in fact, cone points as well. Note that this time the two sides are different parts of the pattern.

4*2: Make sure mirror lines are drawn. Cut along mirror lines, leaving a fundamental square. Mark the gyration point in the middle. Cut a slit from the perimeter of the square to the center point. Overlap the two sides of the slit and form into a cone. Tighten the cone until all 4 vertices converge at one point. These same directions work for 3*3 but starting with a fundamental triangle instead of a square.

*×: Use the stretched-out version of this design from the handout. Cut out a strip of the pattern along two adjacent mirror lines. Bend the strip in the middle and add a half twist. Slide the strip into a tighter and tighter Möbius band until the pattern occurs only once.

**: Cut out a strip along two mirror lines, leaving a third mirror line between them. Fold the strip along this third mirror line. Bend into a cylinder shape and tighten until there is only one copy of the pattern.

Calculating the Costs of Orbifold Signatures

Using Conway's definition for the "cost" of a pattern's signature [2, p. 29], we calculate the "cost" of patterns and collect the results. Thurston [7, Chapter 13, pp. 311-318] defines an Euler number for orbifolds, related to the Euler number from topology, and goes on to prove that a certain class of 17 orbifolds, which he calls "parabolic" are exactly those with orbifold Euler number 0. These correspond to orbifolds for wallpaper patterns. The cost which Conway defines is actually an accounting of this orbifold Euler number. Conway starts with a sphere, with Euler number 2, and adds features until the Euler number is 0.

1. Use the red cards from the card-matching game and assign each participant one or two of them.
2. Use the values in Table 1 to calculate the costs of the signatures associated with each card.
3. Share and compare the results. Notice that the cost of all signatures comes to exactly $2.
4. Discuss how the definitions of cost put arithmetic limitations on the number of possible signatures that come to exactly $2. In fact, there are exactly 17 such signatures [2, pp. 29-41].

Table 1: *Costs of Symbols in an Orbifold Signature [2, p. 29].*

Blue Symbols (orientation preserving)		Red Symbols (reflecting)	
Symbol	Cost ($)	Symbol	Cost ($)
O	2	* or ×	1
2	1/2	2	1/4
3	2/3	3	1/3
4	3/4	4	3/8
5	4/5	5	2/5
6	5/6	6	5/12

Materials

Participants will need colored pencils, acrylic mirrors, rulers, tape, and scissors to construct orbifolds for selected patterns. The supplementary file contains full-sized handouts for personal or classroom use. We created one image that was sampled to create 17 original wallpaper patterns that could be used to print fabrics or personally designed wallpaper for your own house or office (Figure 3). A second set of patterns were created with pattern grids turned on, so mirror lines can be seen (Figure 4).

Figure 5: *A miniature plan for a kimono using our patterns, and one of the authors modeling a completed kimono made from custom-printed fabric featuring all 17 of our wallpaper patterns.*

An Artistic Application

We connected these patterns directly to the paper design for a kimono. John Marshall suggests strategies for designing Japanese clothes that you can make for yourself [4]. A kimono was considered because there is very little waste of material and there are large rectangular regions that lend themselves to surface design.

Some Tools to Use to Develop Your Own Original Patterns

Two powerful tools for playing with patterns are available for iPads. iOrnament [5] by Jürgen Richter-Gebert is a tool that lets you use your fingers or a stylus to draw your own repeating patterns. It is possible to design any of the 17 wallpaper group symmetries and move quickly from one to another. It will also let you explore rosette patterns, patterns on a sphere or even patterns on a polyhedron via an extension called iOrnament Crafter. KaleidoPaint [8] by Jeffrey Weeks is another design tool that works for iOS or Android devices. It also lets you design wallpaper patterns for each of the group symmetries. Centers of rotation and mirror lines can be made visible.

Suggested Extensions Beyond the Scope of this Workshop

Previous Bridges sessions have explored some of these concepts. Vi Hart lead a workshop on snowflakes on a sphere at Bridges 2013 [3]. Manuel Arala Chaves [1] used software to explore symmetry of plane patterns including both wallpaper and frieze patterns in Bridges 2011. Outside of Bridges, Brigitte Servatius has also written a paper called "the Geometry of Paper Dolls" [6]. Based on these ideas as well as a couple of our own, we suggest the following topics and ideas for further exploration of orbifolds and wallpaper symmetry:

- Two color patterns
- Spherical applications
- Orbifold classification of polyhedra
- Proof of the existence of exactly 17 wallpaper groups
- "Wallpaper snowflakes" made by folding paper into orbifolds and cutting
- Cutting folded paper strips into frieze patterns

Acknowledgements

Thanks to Frank Gould for his review and mathematical content advice.

References

[1] M. A. Chaves. "Proposal for a Workshop on Symmetry of Plane Patterns." *Bridges Conference Proceedings*, Coimbra, Portugal July 27-31, 2011, pp. 695-696. https://archive.bridgesmathart.org/2011/bridges2011-695.html

[2] J. H. Conway, H. Burgiel and C. Goodman-Strauss. *The Symmetries of Things*. A.K. Peters, Ltd. 2008.

[3] V. Hart. "OrbiFold and Cut." *Bridges Conference Proceedings*, Enschede, the Netherlands July 27-31, 2013, pp. 635-638. https://archive.bridgesmathart.org/2013/bridges2013-635.html

[4] J. Marshall. *Make your Own Japanese Clothes: Patterns and Ideas for Modern Wear*. Kodansha USA. 2013.

[5] J. Richter-Gebert. *iOrnament*. App available through the Apple App Store, 2012-2018.

[6] B. Servatius. "The Geometry of Folding Paper Dolls." *The Mathematical Gazette*, Volume 81 Number 490, The Mathematical Association, Leicester, UK. 1997.

[7] W. P. Thurston. "The Geometry and Topology of Three-Manifolds." This is an Electronic Version of 1980 notes distributed by Princeton University. March 2002. http://www.msri.org/publications/books/gt3m/

[8] J. Weeks. *KaleidoPaint*. Download available through geometrygames.org. http://geometrygames.org/KaleidoTile/index.html