A Method for Creating Dendritic Fractal Tiles

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Abstract

I describe a process for creating dendritic fractal tiles that admit monohedral two-color tilings. At each iteration of the construction small untiled spaces are present. These decrease in density with each iteration, disappearing in the fractal limit. I present dendritic fractal tiles based on regular square and hexagonal tilings. Some of these tiles are closely related to well-known fractal curves.
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Algorithm for Creating Dendritic Tiles

In Grünbaum and Shephard's book Tilings and Patterns [1] a tiling is defined as a countable family of closed sets (tiles) that cover the plane without gaps or overlaps. Monohedral tilings, in which all the tiles are congruent, tend to be more compelling than those with multiple distinct tiles. Generally, the most appealing tilings are also two colorable, meaning two colors suffice to ensure no adjacent tiles are of the same color. The tilings discussed here have both those properties.

The tiles described here are dendritic fractal tiles, by which I mean tiles with dendritic character (exhibiting tree-like branching) that possess a fractal boundary. The tiles described here tile the plane in periodic fashion. The starting point for their construction is a regular tiling of either squares or hexagons. These constructs only become two-colorable monohedral tilings in the fractal limit.

Fractal tiles that tile in periodic fashion have been described previously, but they are not dendritic [2]. In fact, the Koch snowflake tiles periodically using two different sizes of the tile [3]. I previously described most of the tiles contained in this paper online [4].

A basic procedure for generating these types of tilings follows, with the steps illustrated in Figure 1:
a) Start with a simple periodic tiling, with alternating colored tiles and white "tiles" representing gaps. The colored tiles should be of two colors that also alternate. Regular tilings of squares and hexagons are convenient starting points, but there are other possibilities.
b) Identify a starting grid with small cells, the corners of which are located at the centers of white spaces, and containing at least one white space fully enclosed by each cell. Choose one cell in the grid to serve as a key cell; for example, the bold square in Figure 1b.
c) Fill in the key cells according to some set of rules. The following rules are used for the square-grid examples below. Fill in all the white spaces that are fully contained in the key cell with one color. Then fill white spaces in adjacent cells with the second color, continuing to alternate colors in additional cells of the full grid.
d) Identify a larger grid, in most cases a different orientation, with cells of similar shape, and identify a key cell.
e) Fill in all the white spaces that are fully contained in the new key cell with the same color used previously for the key cell. Then fill white spaces in adjacent cells with the second color, continuing to alternate colors in additional cells of the full grid. Note this step is the second iteration of Step c.
f) Return to step d, treating the larger grid as the starting grid for the next iteration and creating a new larger grid in analogous fashion to the previous iteration, with a consistent relationship between pairs of key cells of successive generations. Repeat steps d and e as many times as desired.

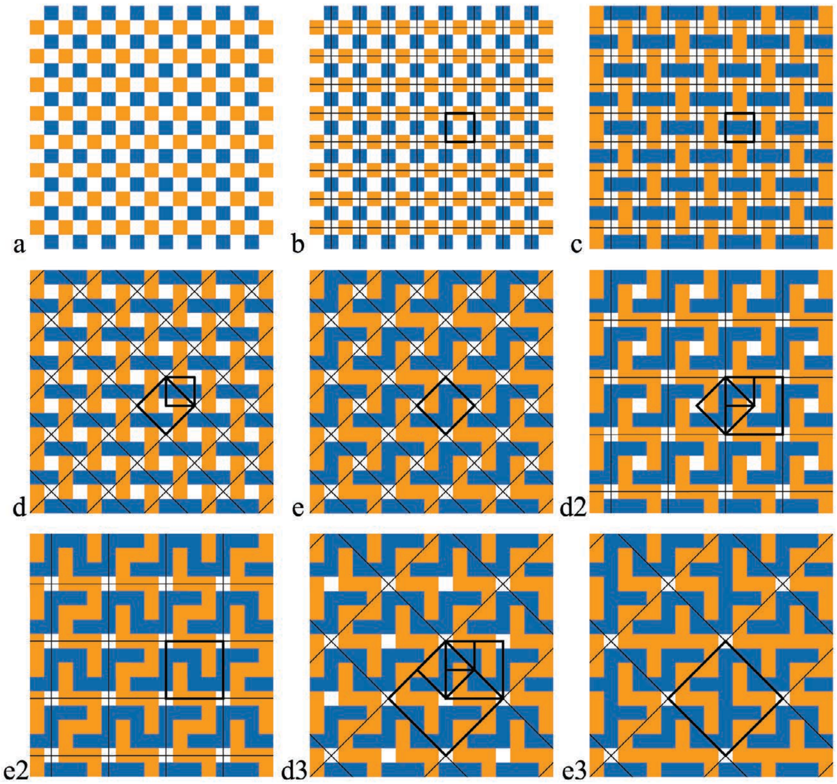


Figure 1: *Initial steps in creating a fractal dendritic tile. The letters correspond to the steps described in the text. The key cells have heavier lines. The fourth and fifth steps are repeated a second time (d2 and e2) time and a third time (d3 and e3).*

With each iteration, the number of tiles decreases, the size of each tile increases, and the number of remaining white spaces decreases. In the limit, the area occupied by white spaces goes to zero, leaving a two-color tiling of fractal tiles. The two-color property stems from the way colors are distributed in Steps a, c, and e above. Four copies of the tile created in Figure 1 after two additional iterations are shown in Figure 2.

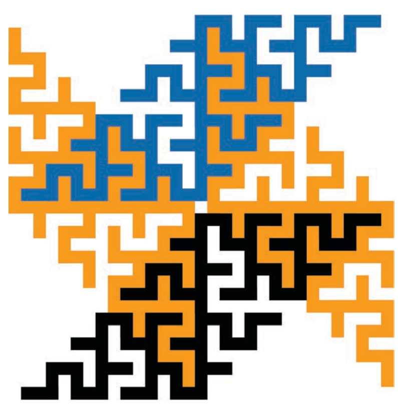


Figure 2: *Four copies of the tile of Figure 1 after two additional iterations.*

If a dendritic tile that is not fractal is desired, a tessellating tile can be formed at any stage by adding triangles to fill the open spaces, as shown in Figure 3 for the tile of Figure 1.

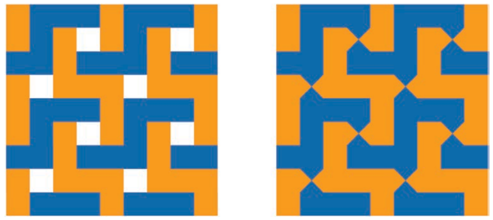


Figure 3: *Dividing each white square into four triangles after any number of generations allows a two-color tiling.*

Additional Dendritic Tiles Based on a Square Grid

In the example above, the key cell scales by the square root of 2 and rotates 45° between successive generations. Given these parameters, the relative arrangement of successive key cells can dramatically change the tile that results. In the first example, the upper left corner of the original key cell is pinned, and the cell alternately rotates 45° counterclockwise and clockwise about that point (Figure 4a).

Using the same scaling factor, an alternative relationship between key cells is illustrated in Figure 4b. In this case, the larger key cell rotates 135° clockwise, and the edge to the left of the arrow is aligned along a diagonal of the smaller square. This rule results in a tile that is essentially the twindragon curve [3], as shown in Figure 5.

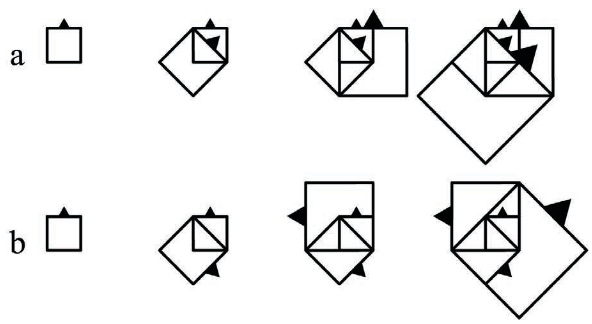


Figure 4: *a. The rule for mating key cells used in Figure 1. b. An alternative rule for mating key cells from generation to generation.*

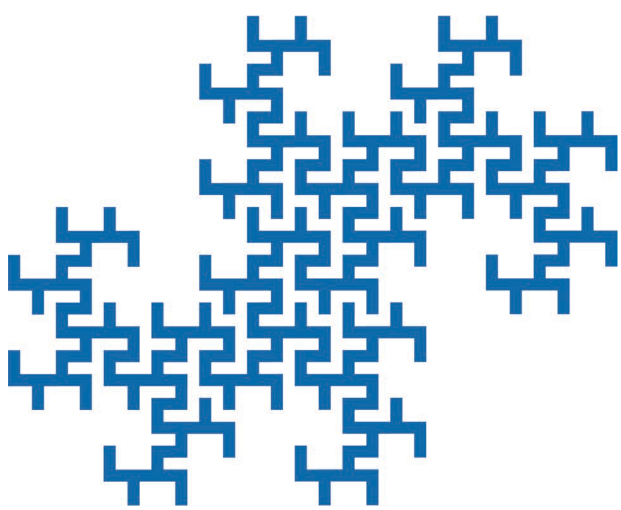
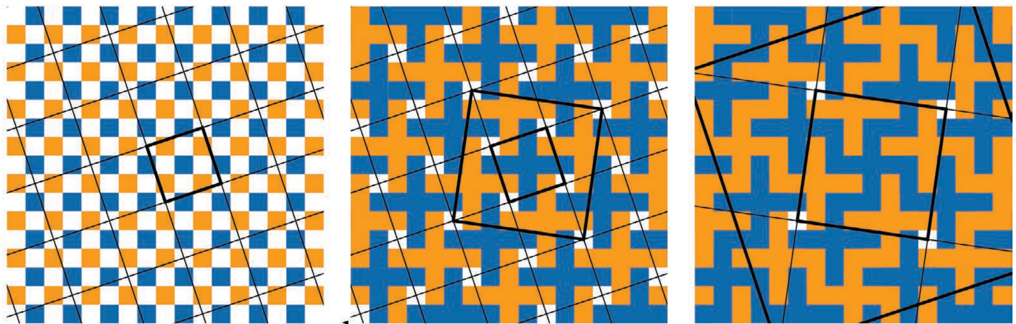


Figure 5: *The tile created after eight iterations according to the rule of Figure 4, which is essentially the twindragon fractal.*

Larger scaling factors can be used as well. In Figure 6, the starting key cell encloses four white squares. Filling those connects five darker-colored squares. The next-generation key cell is scaled up by the square root of five, as shown in Figure 6b. Filling the white squares in it connects five crosses, as shown in Figure 6c. After two more iterations, the tile of Figure 7 is obtained. Note this tile has four-fold rotational symmetry, while those above have two-fold symmetry.



a b c

Figure 6: *An example in which the scaling factor between generations is the square root of 5.*

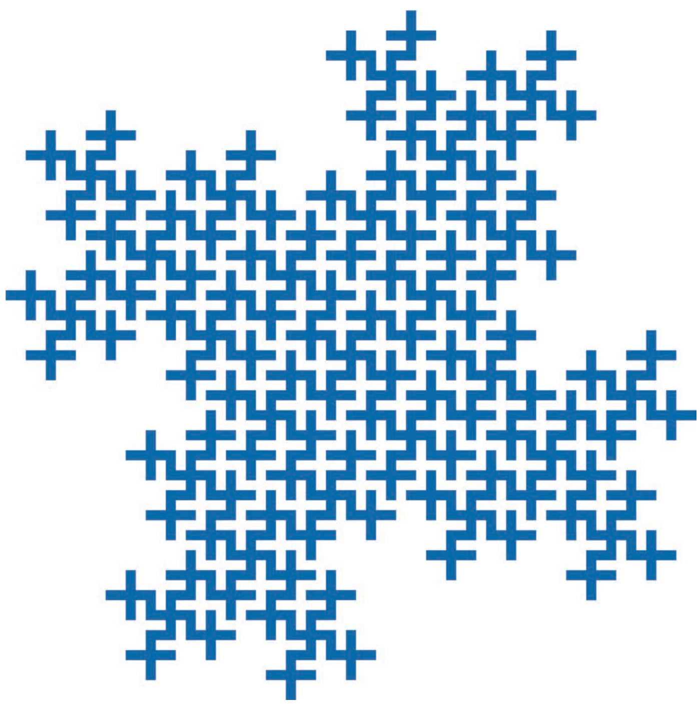


Figure 7: The tile of Figure 6 after four iterations.

Dendritic Tiles Based on a Hexagonal Grid

Another convenient starting grid is the regular hexagon tiling. In the examples shown here, 1/3 of the hexagons are one color, 1/3 a second color, and 1/3 white (representing blank spaces). The simplest starting key cell is an equilateral triangle that encloses a single white hexagon. If the next generation key cell is scaled by the square root of three and rotated 30 degrees clockwise about a corner of the starting key cell, the tiles of Figure 8b result. The next iteration is shown in Figure 8c. Each time a hexagon is filled three tiles are joined, and the tiles have three-fold rotational symmetry. After two more iterations the tile of Figure

9 is obtained, where six copies are shown together. This tile can be also obtained by joining three copies of a hexagon version of the terdragon curve [3]. Ignoring the penetrating channels, its boundary is a fractal called the fudgeflake, which also tiles the plane [3].

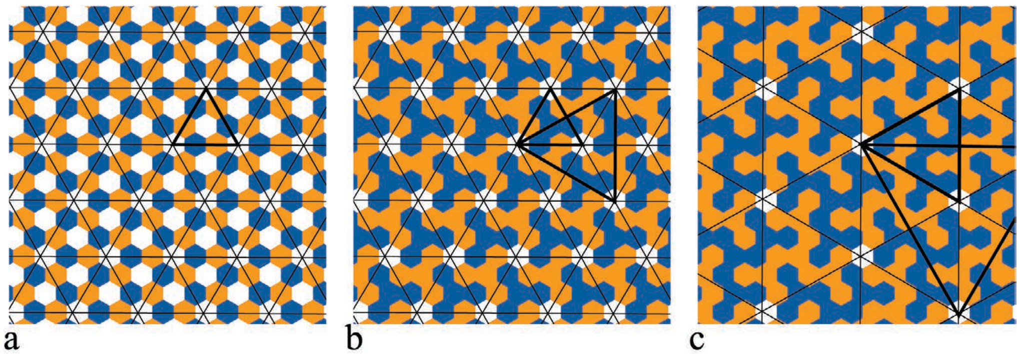


Figure 8: *Creation of a dendritic tile from a grid of hexagons. The grid scales between successive generations as the square root of 3.*

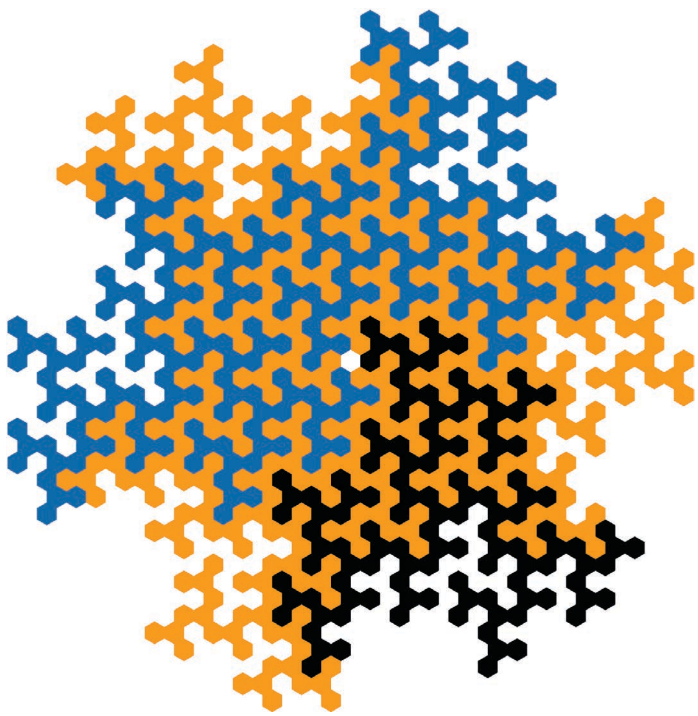


Figure 9: *Six copies of the tile of Figure 8 after two additional iterations. One of the darker-colored tiles is colored black here to make it easier to see a single tile.*

A larger starting key cell is shown in Figure 10a, enclosing three white hexagons. Filling those joins seven darker-colored hexagons. The key cell scales by the square root of seven between successive generations, creating a tile with three-fold rotational symmetry. The tile has mirror symmetry after one iteration, but not after additional iterations, as seen in Figure 10c. The tile resulting from a third iteration is shown in Figure 11. Ignoring the penetrating channels, this tile has the same boundary as the Gosper Island, which also tiles the plane [3]. As in the square case, if a dendritic tile that is not fractal is desired, a tessellating tile can be formed at any stage by adding triangles to fill the open spaces, as shown in Figure 12 for the tile of Figure 10.

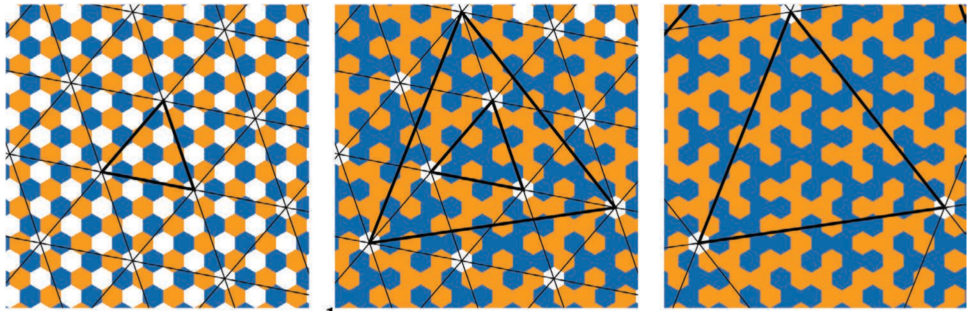


Figure 10: *An example in which the scaling factor between generations is the square root of seven.*

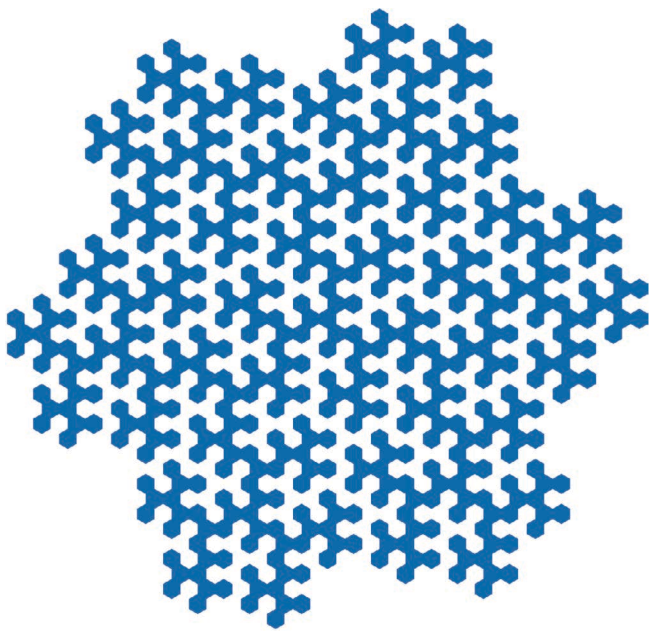


Figure 11: *A single tile generated by the rules of Figure 10, carried through a third iteration.*

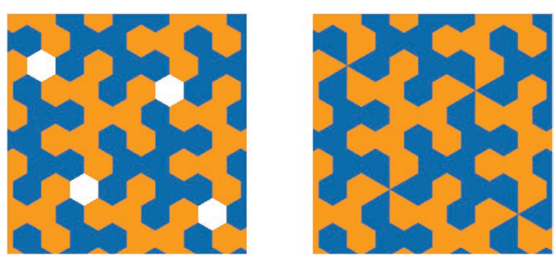


Figure 12: *Dividing each white hexagon into six triangles after any number of generations allows a two-color tiling.*

A wider variety of tiles can be generated if the construction rules are relaxed. One possibility is to color the starting tiling differently. In Figure 13a, instead of starting from alternating colors, a coloring is used whereby all the colored tiles in each cell of the starting grid have the same color. This leads to first-iteration tiles that approximates an equilateral triangle. Subsequent iterations keep these groups separate other than single joining hexagons. This results in a tile (Figure 14) that is less dendritic in appearance but has the interesting property of looking at a glance like a patch of the regular triangle tiling.

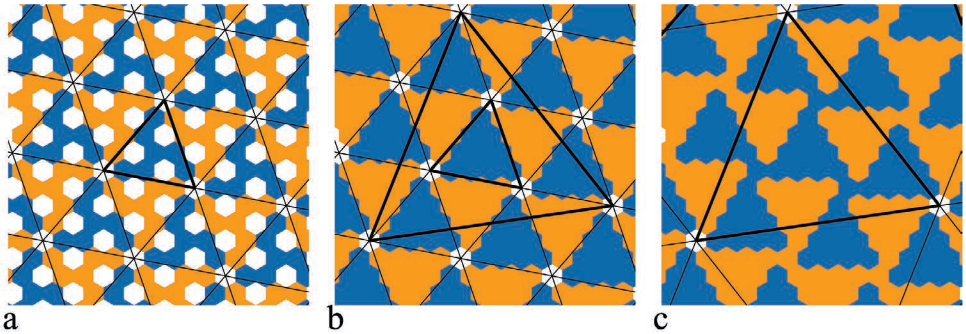


Figure 13: *An example starting from a different coloring of the tiles in the initial tiling.*

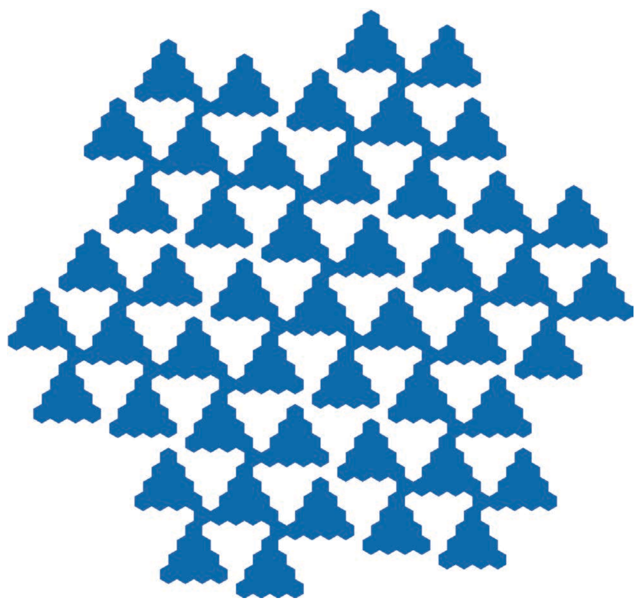


Figure 14: *A single tile generated by the rules of Figure 13, carried through a third iteration.*

Stepped Structures

Three-dimensional structures can be formed by converting the individual building blocks of the curves to prisms. This allows the creation of stepped structures on which one can imagine walking [5]. A particular iteration of the twindragon dendritic tile is shown in Figure 15, where the squares at the ends of the curves are the lowest height steps and the center square the highest step. Four of these structures are fit together in Figure 16 as they would be in the tiling.

Summary and Conclusions

I presented a method for designing dendritic tiles admitting monohedral two-color tilings in the fractal limit. I gave examples based on the regular square and regular hexagon tilings. Many additional tiles are possible by using different construction rules. The iterative construction process yields beautiful and complex tiles, the fitting together of which almost appears magical. Dendritic tiles of this sort carried through a finite number of iterations could be employed in puzzles. In principle this method could be extended to three dimensions, allowing the creation of three-dimensional space-filling dendritic fractal tiles.

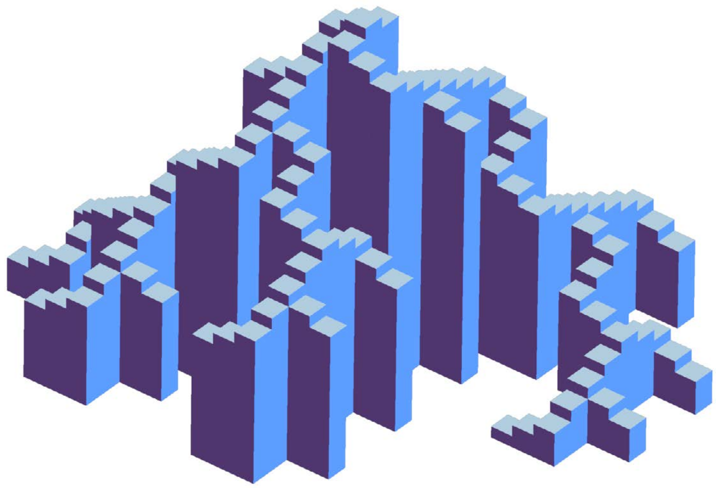


Figure 15: *A stepped structure formed from the sixth iteration of the twindragon tile shown in Figure 5.*

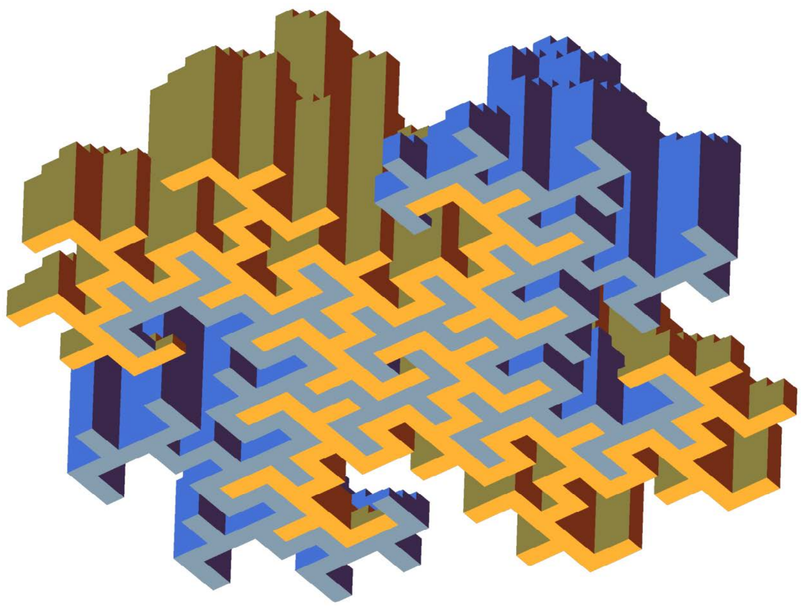


Figure 16: *Four of the tiles of Figure 15 fit together, seen from the bottom side.*

References

[1] B. Grünbaum and G.C. Shephard. *Tilings and Patterns*. W.H. Freeman, 1987.

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[5] R.W. Fathauer. "Walkable Curves and Knots." *Bridges Conference Proceedings,* Linz, Austria, 2019, pp. 13–20. http://archive.bridgesmathart.org/2019/bridges2019-13.html.