Eight Heptagons: The Double Torus Revisited

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Abstract

A decade ago, in search of a versatile eight-color map on a double torus, I discovered an intriguing gluing of eight heptagons. Topologically, the glued heptagons form a surface of genus two on which each heptagon shares a border with each of the others, demonstrating that at least eight colors are required to color an arbitrary map on a double torus. While I used this configuration of heptagons to create an eight-color ceramic tea set in 2010 and a bead-crochet eight-color double torus in 2014, both of these designs significantly deformed the eight regions into various shapes conforming to each surface. This paper shows what happened when I stitched together eight congruent crocheted heptagons into the double-torus map, an endeavor I first attempted this year. The result is a delightfully twisted fabric model with the two holes in roughly perpendicular directions.

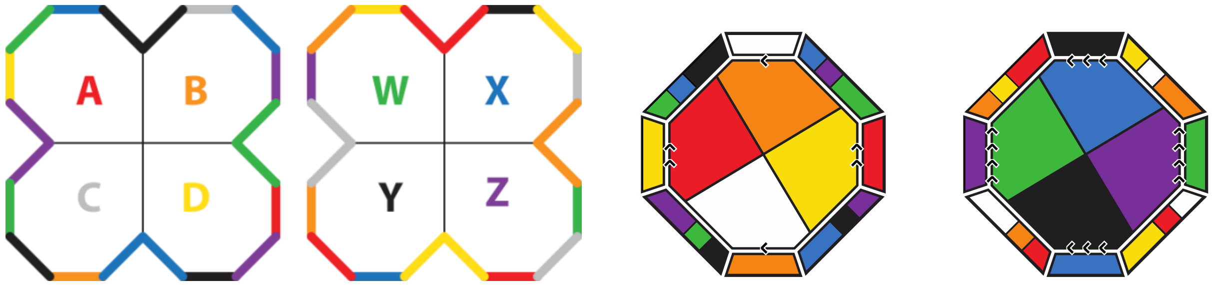
The Heptagon Schematic

In 1890, P.J. Heawood proved that eight colors are sufficient to color any map on a double torus and ensure that countries with a common border have different colors [3]. To establish the need for eight colors, it suffices to construct a double-torus map with eight countries, each of which touches all of the others. Various mathematicians and artists have taken up the challenge [2,3,4]; Figure 1 shows two of my artworks on this theme, the painted tea set *Tea for Eight*, and the bead crochet pendant *Eight-Color 8*.

(a) (b)

Figure 1: *Artworks featuring double torus maps: (a)* Tea for Eight*, 2010, (b)* Eight-Color 8*, 2014.*

In my Bridges 2014 paper, I described how I created the maps for the tea set and pendant by finding a gluing of eight heptagons that has the Euler characteristic of a genus two surface [4]. Figure 2a shows a rendering of the diagrams I originally drew by hand in my search for the edge connections that would give me the surface I wanted. However, the diagrams that I actually included in the 2014 paper and used to design my artworks were those in Figure 2b, in which each cluster of four heptagons has been rearranged into an octagon. This streamlined configuration makes it clearer that each cluster forms a punctured torus, and the two punctured tori are joined at a seam to produce a double torus. It also clarifies a practical observation: the critical part of drawing this map on a preexisting two-holed surface is arranging the proper color contacts around a seam between two halves of the surface [4]. Once that is achieved, it is straightforward to get each set of four colors to touch on each punctured torus, and an artist has considerable freedom to wind those colors around the natural contours of the artistic medium.



(a) (b)

Figure 2: *The gluing of eight heptagons from my 2014 Bridges paper: (a) a rendering of the diagram from my original notes, (b) the version I put in the paper [4] and used for the works in Figure 1.*

Having extracted this essential information, I forgot about the heptagons that gave rise to it until another experiment in crocheted maps brought me back to my starting point.

Seven Crocheted Hexagons

My interest in double-torus maps was ignited by the single-torus analog, a map on a torus with seven countries, each of which touches all of the others. The knowledge that a map of this kind results from tiling the plane with regular hexagons, applying a periodic seven-coloring to the tessellation, and gluing together the edges of a fundamental region is what led me to the eight-heptagons attack on the double torus map. For a long time, I had convinced myself that sewing together actual hexagons to make a fabric torus map required deforming the hexagons so that they were no longer regular. More recently, conversations with colleagues and the discussion of the hexagon-based invertible toroidal scarf in Ellie Baker and Charles Wampler's Bridges 2017 paper [1] showed me the error of my thinking, and I planned to hand sew the corresponding torus out of fleece hexagons when time permitted.

Then it occurred to me that the traditional granny square motif in crochet had already been adapted into other regular polygons, including the granny hexagon, and crocheting a regular hexagon is less tedious and more satisfying than measuring and cutting hexagons from fabric. I promptly pulled seven colors of yarn from my yarn stash and made a torus similar to the one in Figure 3. The torus pictured here is *Granny's Torus*, made from nicer yarn acquired for the purpose. I soon discovered that among her many wonderful topological creations, Moira Chas had crocheted a similar torus with a different regular-hexagonal motif several years earlier [2].

(a) (b) (c) (d)

Figure 3: *Four views of* Granny's Torus. *Views (a) and (b) opposite sides of the cylinder configuration, and views (c) and (d) are opposite sides of the triangle configuration.*

For Bridges 2011, John Sullivan wrote about the geometry of a number of tiled tori, including the type shown here [6]. His paper predicts the wavy structure visible in Figures 3a and 3b, in which the torus is arranged so that the outermost surface is roughly cylindrical. Like the invertible scarves in the

Baker/Wampler paper [1], *Granny's Torus* can also be folded into an equilateral triangle as shown in Figures 3c and 3d, though the thickness of crochet makes the folding slightly messier than in a silk scarf.

After I completed my first granny hexagon torus, it suddenly occurred to me that there was nothing stopping me from crocheting a heptagon…

Eight Crocheted Heptagons

Each of the heptagons in Figure 2 has one vertex that meets three other heptagons, while its remaining vertices meet two other heptagons. This suggests that the optimal crochet interpretation of the heptagon will have one right angle and six 120° angles. To obtain this, I grafted together the patterns for a granny hexagon and a solid granny square into the motif shown on the right of Figure 4. The interior angle sum being less than 900°, the resulting granny heptagon is negatively curved and will not lie flat. It is somewhat reminiscent of the purely hyperbolic granny polygons that Joshua Holden and Lana Holmes Holden developed [5], although their motifs have uniform negative curvature and incorporate much more sophisticated geometry.

Figure 4: *Each heptagon (right) is a combination of a granny hexagon (lower left) and one quarter of a solid granny square (upper left). The resulting polygon has the six 120° angles of the hexagon and the one 90° angle of the quarter-square; the square corner of the green heptagon is in the upper left.*

Crocheting the heptagons was a bit of a leap of faith, as I was genuinely unsure of whether sewing the parts together was physically possible. Indeed, it took me a partial (and in retrospect, probably unnecessary) disassembly and some reconstructive surgery to get all of the edges sewn together. Once I had assembled the eight polygons that match the heptagon in Figure 4 and played with it a bit, I decided that I needed to make a bigger double torus that was easier to photograph and manipulate. The result, *Granny's Double Torus*, appears in Figure 5.

A key surprise of the crocheted double torus was the discovery that the two holes are oriented in roughly perpendicular directions. The photographs here show the model in its most comprehensible configuration, with the degree-four vertices at the top and bottom (Figures 5a and 5b). Figure 5c is a side view looking through the hole closer to the vertex touching red, orange, yellow and white. Turning the double torus a quarter turn so that the top moves to the left gives the view in Figure 5d, looking through the hole closer to the vertex touching green, blue, purple, and black. Figure 5e gives a separate illustration of the orientation of the two holes.

To readers who crochet and enjoy topology, I strongly recommend grabbing some yarn and making your own double-torus map. The assembly process is a fun puzzle in itself, but if you prefer clearer guidance, the supplementary file in the Bridges archive gives step-by-step assembly instructions. As it happens, Granny has a lot to teach us about topology.

(a) (b)

(c) (d) (e)

Figure 5: Granny's Double Torus: *(a) one vertex of degree 4, (b) the other vertex of degree 4, (c) the hole on the right of the first photo (front side), (d) the hole on the left of the first photo (back side), (e) a topologist studies the properties of* Granny's Double Torus.

Acknowledgements

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