Optimizing Morton's Tritangentless Knots for Rolling

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Abstract

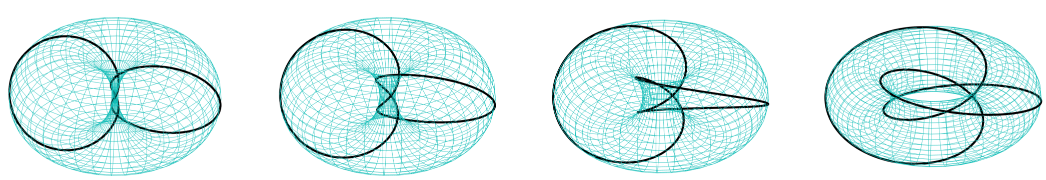
Tritangentless knots have a curious and beautiful property: when realized as physical 3D printed models, they *roll*. Some tritangentless parameterizations roll more easily and freely than others. In this paper we numerically optimize parameters to obtain the most "aesthetically pleasing" rolling knots and then create physical models of these knots using 3D printing, thereby leveraging mathematical tools to obtain an elegant kinetic sculpture.

Visualizing Morton's Knots

A closed curve in $\mathbb{R}^3$ is said to be *tritangentless* if there are no planes which are simultaneously tangent to three distinct points on the curve. In 1978, Freedman [1] conjectured that any closed curve in $\mathbb{R}^3$ without tritangent planes must be unknotted. This conjecture was proved false in 1991 when both Morton [4] and Montesinos [2] exhibited explicit parameterizations of trefoil knots without any tritangent planes. Morton's parameterization is:

$$x(t) = \frac{ca\cos(3t)}{1 - b\sin(2t)}, \quad y(t) = \frac{ca\sin(3t)}{1 - b\sin(2t)}, \quad z(t) = \frac{cb\cos(2t)}{1 - b\sin(2t)}, \quad \text{for } 0 \le t \le 2\pi, \tag{1}$$

with $a^2 + b^2 = 1$, where c is a constant we have added for scaling. Although Morton's paper included this explicit algebraic parametrization it did not include any illustrations of these interesting knot conformations. In this work we will optimize the shape/style parameter a and an additional z-scaling factor to identify the Morton knots that roll most efficiently, and then visualize those knots in three dimensions by creating 3D printed models that can be physically rolled on flat surfaces.



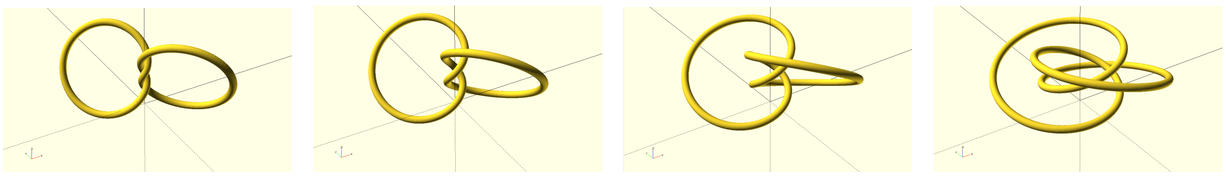
(a) $a = 0.3$ **(b)** $a = 0.5$ **(c)** $a = 0.7$ **(d)** $a = 0.9$

Figure 1: *Morton knots on tori with $R + r = 1$ and various a values.*

Morton's trefoil is a $(3, 2)$ torus knot that we can think of as embedded on a torus with major radius R and tube radius r. To best compare visualizations of Morton knots we will fix $R + r$, so that we can consider conformations with different values of a embedded on tori that shadow the same circle in the x-y plane. For example, in Figure 1 we set $R + r = 1$ and consider torus embeddings of Morton knots with $a = 0.3, 0.5, 0.7,$

and 0.9. In this example we have $R = 1/(1+b)$, $r = b/(1+b)$, and $c = a/(1+b)$, and the listed values of a correspond to radii of $(R, r) \approx (0.5118, 0.4882)$, $(0.5359, 0.4641)$, $(0.5834, 0.4166)$, and $(0.6964, 0.3036)$.

Figure 1 makes it clear that different values of a result in very different knotted shapes, some of which are likely to roll better than others. In order to physically test the rolling properties of these knots we must have three-dimensional realizations of these conformations with nonzero tube thickness; we keep this thickness as small as possible so as to stay close to the idealized mathematical model. Figure 2 shows models of tubified Morton knots that we created with OpenSCAD "sweeper" code [6].



(a) $a = 0.3$ **(b)** $a = 0.5$ **(c)** $a = 0.7$ **(d)** $a = 0.9$

Figure 2: *Three dimensional visualizations of Morton knots with $R + r = 1$ and various a values.*

Rolling Knots

By definition, a tritangentless knot conformation cannot include three points that share a tangent plane. Considering such a knot as a physical closed curve in space that is sitting on a flat surface, being tritangentless means that the curve can never have more than two points touching the flat surface at any given time. This is what makes models of tritangentless knots roll.

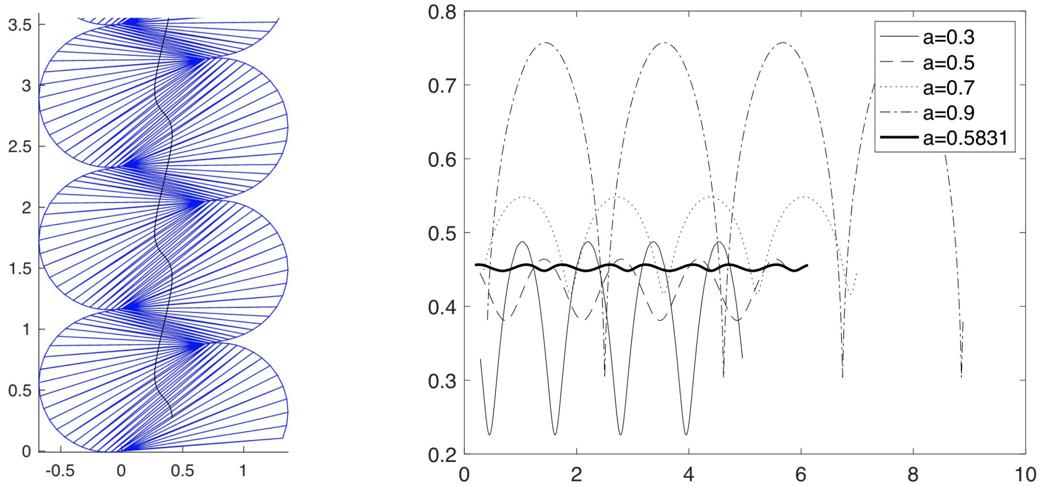
We can get an idea of how a knot rolls across a flat surface by using a convex hull approximation. Start by approximating the knot with a set of data points based on equally spaced values of t on $[0, 2\pi]$, and then approximate the convex hull from these data points, resulting in a collection of long, thin triangles. As the number of data points increases, the triangles become increasingly thin, and in the limit become lines whose endpoints lie along the path of contact. We can associate with each triangle's two long edges the adjacent long thin triangles along those two edges; following these triangles in order gives the order in which they touch the plane as the knot rolls. We can then map the triangles to the plane, rotated and shifted so that the knot is on average moving in the direction of the y-axis, starting with one point at the origin. For example, Figure 3(a) shows the part of the plane traversed by rolling Morton's knot with $a = 0.3$, $R + r = 1$ through about one and a half rotations. The central meandering line shows the position of the center of mass mapped down to the plane. The center of mass's position can be calculated relative to the location of each long thin triangle on the plane.

The Rolliest Morton Knot

The center of mass of a Morton knot is at its center (which due to its symmetry is equidistant from its "top" and "bottom" in any orientation), and this center rises and falls as the knot rolls across a flat surface. The smaller the vertical variation of this center of mass, the less energy needs to be transferred between kinetic and potential, which results in a smoother and longer roll. As we vary the parameter a in equation (1) we obtain different amplitudes of oscillation in the vertical direction of the center of mass, as shown in Figure 3(b).

Note that for values of a at the extremes, the center of mass changes height quite dramatically. Such large changes make it very difficult for the knot to rotate, as they require the center of mass to go "uphill" a substantial distance. 3D printed models of those knots won't roll unless we physically push them.

To find the best value of a for rolling, we need to look for the value with the smallest variation in the



(a) *Side-to-side position of center of mass as the knot rolls across a surface*

(b) *Height of the center of mass (vertical axis) as a function of movement direction.*

Figure 3: *Center of mass movement from side to side for one Morton's knot with $a = 0.3$ and $R + r = 1$ (a), and up and down for many Morton's knots with differing values of a and $R + r = 1$ (b).*

vertical direction, scaled by the average height of the knot. Specifically, for every value of a we can "roll" the knot as we did in the previous section, identify the difference between the maximum and minimum heights of the center of mass, and divide by the average center of mass height. We can either plot this measure of vertical variation and successively zoom in and recalculate near the minimum to identify the best rolling knot, or simply use minimization software. Since a derivative cannot be explicitly calculated here, we use the standard Nelder-Mead simplex method [5] in Matlab. We also need to ensure that there are a sufficient number of points in the initial point set; experimentation suggests that around 800 points are more than sufficient.

Using these numerical methods we find that the best-rolling Morton knot has the parameter $a = 0.5831$, shown as the solid line in Figure 3(b). For this value of a, the range of movement of the center of mass is just 0.01864, that is, less than 2%. For comparison purposes, the center of mass variations for $a = 0.3, 0.5, 0.7,$ and 0.9 are 0.8316, 0.1972, 0.2702, and 0.8738, respectively.

A Family of Optimized Morton Knots

In the previous section we identified the "best" Morton knot for rolling, but in fact we can optimize *every* Morton knot for rolling if we multiply by a z-scaling factor, effectively making the cross section of the embedding torus an ellipse. Scaling a Morton knot parameterization in the vertical direction varies the center of mass locations when the knot rolls. For any given value of a we can choose the z-scaling factor that minimizes the vertical variation of the center of mass.

Figure 4 shows 3D printed models of four optimized Morton knots, for $a = 0.3, 0.5, 0.7,$ and 0.9, with optimized z-scale factors of $0.4629, 0.8210, 1.316,$ and 2.495, respectively. The center of mass variations for these knot models are very small; just $0.0047, 0.0134, 0.0285,$ and 0.0615, respectively. All four of these models roll effortlessly given the slightest breeze or incline, but as a increases the side-to-side motion of the center of mass also increases. However, we find that the most aesthetically and kinetically pleasing of these optimized Morton knots is the one with $a = 0.5$, due to its overall symmetric motion.

(a) $a = 0.3$, *stretch* 0.4629 **(b)** $a = 0.5$, *stretch* 0.8210 **(c)** $a = 0.7$, *stretch* 1.316 **(d)** $a = 0.9$, *stretch* 2.495

Figure 4: *3d printed stretched Morton knots with $R + r = 1$ and various values of a.*

Conclusion and Future Directions

This project was initially motivated by a desire to visualize Morton's tritangentless knots. We found not only the best-rolling standard Morton knot, but a method for optimizing any Morton knot for rolling. We also 3D printed physical models of these optimized knots – and they certainly roll excellently! In future work we could investigate whether or not it is possible to get a perfectly idealized rolling Morton knot with no center of mass motion at all.

There are a number of other future directions for this work. First, it may also be worth doing a similar analysis for the tritangentless knot parameterization in [2]. Second, we could generalize Morton's conformation to $(p, 2)$-torus knots for $p > 3$; this would sacrifice the overall tritangentless property (see [3]), but many of these knots still appear to be *externally* tritangentless, in the sense that they still roll. Finally, as we saw in Figure 3(a), convex hulls of Morton knots are examples of *developable surfaces*, meaning that they can be flattened onto a plane without deformation. Future work could investigate other knot conformations with convex hulls that are similar to developable surfaces such as oloids and sphericons.

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