

## Exploring Noble Polyhedra With the Program Stella4D

Ulrich Mikloweit

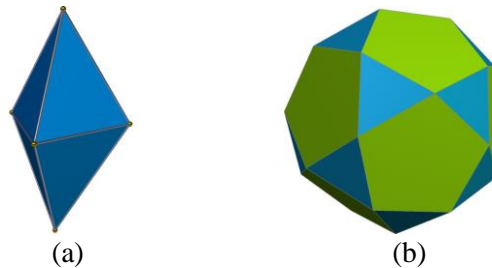
Essen, Germany; [polyart@arcor.de](mailto:polyart@arcor.de)

### Abstract

This paper shows how noble polyhedra can be produced and investigated with the software Stella4D. In a very easy way the results of Max Brückner can be reproduced and verified and many previously unpublished noble solids could be found.

### Introduction

Polyhedra consisting of only one type of faces are called isohedral, and those consisting of only one type of vertices are called isogonal. An example for the first category is a triangular bipyramid, one for the latter category is the icosidodecahedron, see Figure 1.

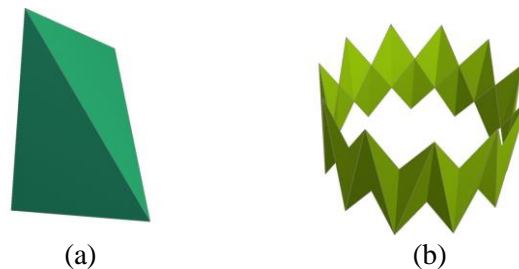


**Figure 1:** (a) *Triangular Bipyramid*, (b) *Icosidodecahedron*.

If both properties occur in the same shape this is called a noble polyhedron.

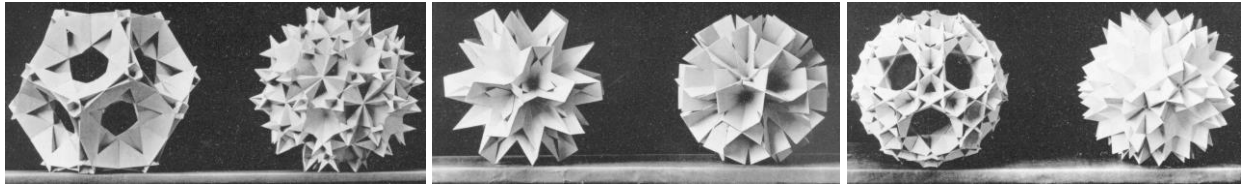
The most well-known noble polyhedra are the five Platonic solids. They are convex and they have the additional property that their faces are all regular. There is only one additional shape known which is both noble and convex, or more accurately: an infinite continuum of distorted shapes. These are the disphenoids. You can create a disphenoid by stretching a tetrahedron along a 2-fold axis. You get an irregular tetrahedron made of isosceles triangles where all faces and vertices are the same (Figure 2 (a)). As you can vary the degree of stretching, there are infinitely many of them.

There are also nonconvex noble polyhedra, with intersecting faces. Among these are the four Kepler-Poinsot polyhedra the faces of which are regular too. One of them has equilateral triangles, one has regular pentagons and two of them are set up by regular pentagrams. The family of nonconvex noble polyhedra also contains an infinite set: starting from a prism or antiprism you can construct an unlimited variety of crown polyhedra, the faces of which are crossed quadrilaterals, see Figure 2 (b).



**Figure 2:** *Two infinite sets of noble polyhedra: (a) Disphenoid, (b) Crown Polyhedron.*

All these were already investigated more than 100 years ago. Based on the works of J. Hessel (Marburg, 1871) [6] and E. Hess (Kassel, 1876) [5], Max Brückner published a book in 1906 [1], where he described many more noble polyhedra and their construction and he even presented pictures of beautiful paper models he built. Most of the polyhedra Brückner investigated are compounds, mostly of disphenoids or crowns, but also of great dodecahedra ([1], plate 26,4) or Stellated dodecahedra ([1], plate 26, 10). In 2009 the book was translated into English by Richard Stratton, Ilse Stratton and me [2]. Unfortunately Brückner's models are lost and trying to reconstruct them according to the pictures is very hard (Figure 3).



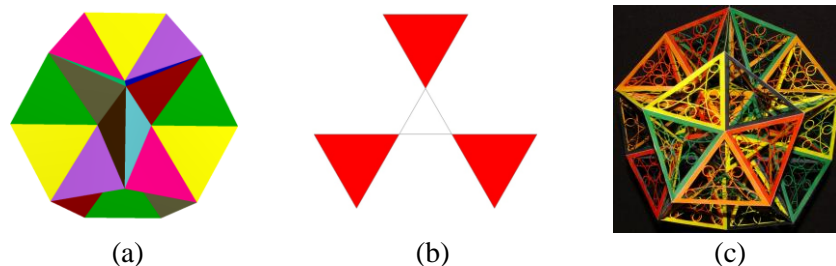
**Figure 3:** Max Brückner [1], Plate 28 (processed).

But we have mighty tools today. With the superb software Stella4D by Robert Webb [7] it is very easy to construct polyhedra of every kind. The program has many cunning features like “Stellation”, where you can expand the faces of a given shape and create a new polyhedron. Another is “faceting” which allows you to connect the vertices of a given polyhedron by new “edges” which produces the faces of a new polyhedron if you do it in a suitable way. I used this technique to search for noble polyhedra. With the faceting tool you can step through facetings by a simple hit on the keyboard, thus producing thousands and thousands of facetings. You can set criteria for these and I chose “isohedral facetings only”. I took the uniform polyhedra as starting points because they are isogonal. Here I was only interested in distinct polyhedra, not in compounds. Beyond the formerly mentioned regular noble polyhedra plus the disphenoids and the crowns, I thus found 52 more isohedral polyhedra which are isogonal too. Of these, 24 were already depicted in Brückner’s book. All these shapes are listed in the Table 1 below. Nineteen of them are previously unpublished as far as I know.

In the following, some of my findings will be described shortly by naming the source polyhedron (characterized by its Wenninger number) from which they were derived [9] and the faces of which they are made. Since noble polyhedra are both isohedral and isogonal, and these are dual properties, the dual of a noble polyhedron will also be noble. So most appear as pairs of duals, but some of them are self-dual. Some have reflection planes and some are chiral. Some look rather weird, as do some of the faces. Some noble polyhedra arise from more than one uniform polyhedron. All these doublets were ignored. In the table you'll find figures and related information.

### Some examples

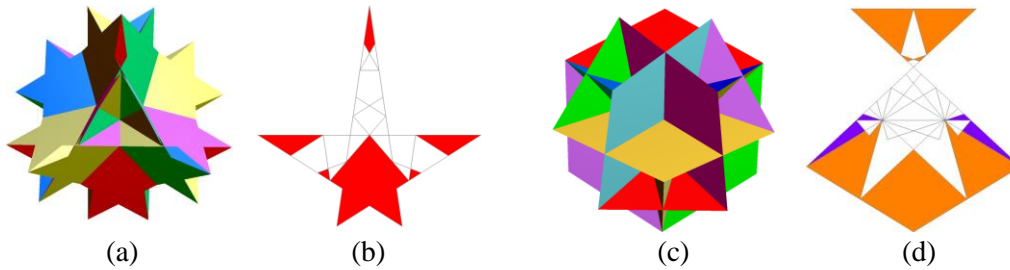
From the dodecahedron (W5) you can derive 7 noble polyhedra. The first one is made of 60 isosceles triangles, 9 of which meet at each vertex. So the 20 faces of its dual are 9-gons. The third one (Figure 4) is an icosahedron too, made of crossed 6-gons. It is self-dual:



**Figure 4:** (a) W5\_02, (b) face of W5\_02, (c) paper model of W5\_02.

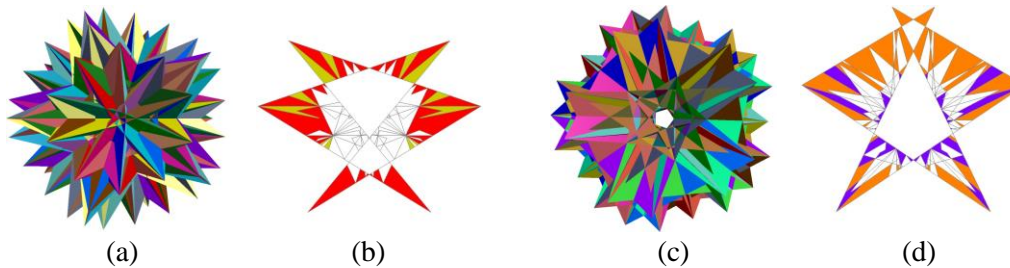
The faces of the next one are 60 crossed 4-gons, and those of its dual are 12-gons. Then you get another hexecontahedron made of isosceles triangles again and its dual is a spiky shape of 9-gons which is the final Stellation of the icosahedron.

With my Stella4D version 5.4, two noble polyhedra could be derived from the truncated cube (W8). The first is a 24-hedron the faces of which are 5-gons (Figure 5 (a) and (b)). Its dual has 24 faces too, and they are also pentagons (Figure 5 (c) and (d)). Among Brückner's models there are two more, the convex hull of which is the truncated cube. These and their duals are given in Table 1.



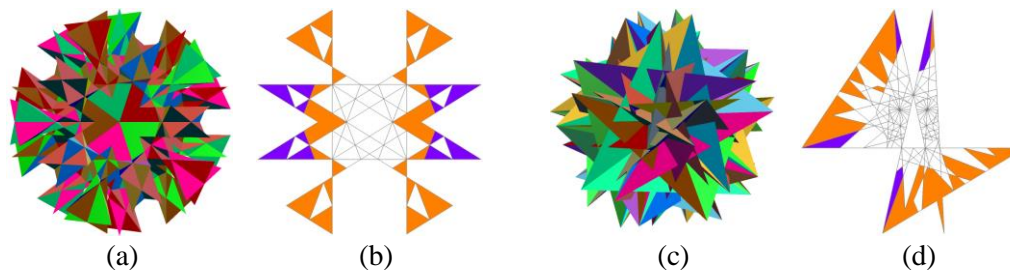
**Figure 5:** (a)  $W8_{01}$ , (b) *face of*  $W8_{01}$ , (c)  $W8_{01_d}$ , (d) *face of*  $W8_{01_d}$ .

The truncated dodecahedron (W10) supports two noble polyhedra, as well. Both have 60 faces and these are multiple crossed 6-gons (Figure 6).



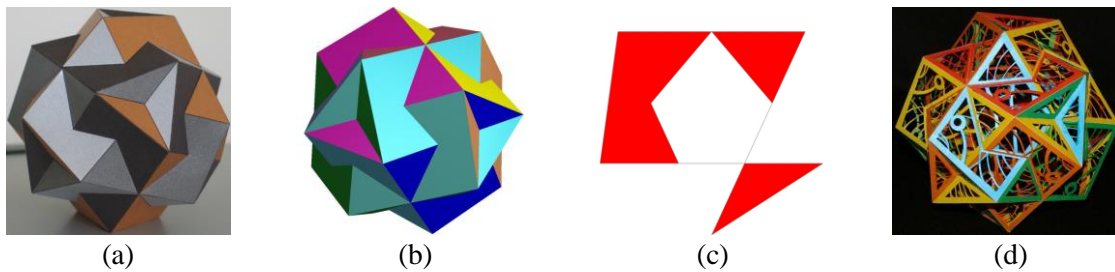
**Figure 6:** (a)  $W10_{01}$ , (b) *face of*  $W10_{01}$ , (c)  $W10_{01_d}$ , (d) *face of*  $W10_{01_d}$ .

A greater number of noble polyhedra comes from the icosidodecahedron (W12). Twelve shapes are available, among these there are 6 triacontahedra, one of them being the final stellation of the rhombic triacontahedron, 3 are hexecontahedra and 3 are hexakisicosahedra. Brückner mentioned already 8 of them. The faces differ from 3-gon to 12-gon. One example is given in Figure 7 (a) and (b). From the rhombicosidodecahedron (W14) there could be derived 10 noble polyhedra. All are hexecontahedra. Crossed pentagons form the faces of six of these, while the other ones are made of 6-gons (Figure 7 (c) and (d)). Two of these models were discussed by Brückner.



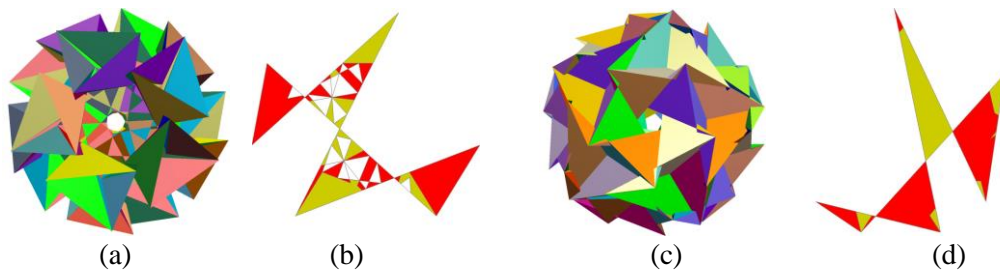
**Figure 7:** (a)  $W12_{04_d}$ , (b) *face of*  $W12_{04_d}$ , (c)  $W14_{03_d}$ , (d) *face of*  $W14_{03_d}$ .

A very nice case is the noble polyhedron which can be obtained from the snub cube (W17). Robert Webb found it in 2008 [8]. It is made of 24 crossed pentagons and it is also self-dual (Figure 8).

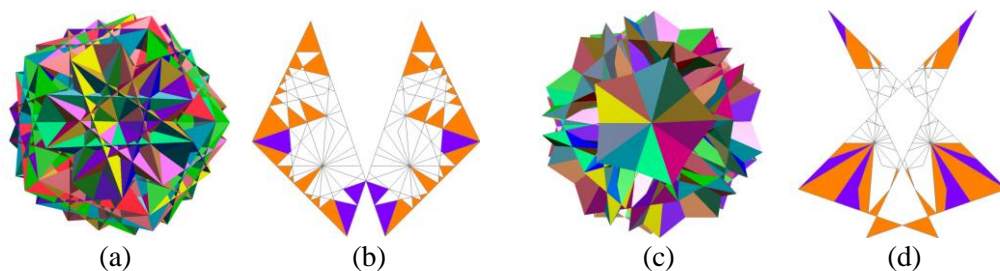


**Figure 8:** (a) paper model of  $W17_{01\_sd}$  by R. Webb, (b)  $W17_{01\_sd}$ , (c) face of  $W17_{01\_sd}$ , (d) paper model of  $W17_{01\_sd}$  by me.

The Rhombicosahedron (W96) provides one self-dual noble polyhedron made of 60 hexagons. Three of the uniform nonconvex snub polyhedra (W110, W112 and W115) enable you to form 8 supplementary noble polyhedra. Their faces are always hexagons and all are hexecontahedra (Figures 9 and 10).



**Figure 9:** (a)  $W110_{01}$ , (b) face of  $W110_{01}$ , (c)  $W112_{01\_sd}$ , (d) face of  $W112_{01\_sd}$ .



**Figure 10:** (a)  $W115_{01\_d}$ , (b) face of  $W115_{01\_d}$ , (c)  $W115_{02\_d}$ , (d) face of  $W115_{02\_d}$ .

Seven models of distinct polyhedra Brückner describes could not be reproduced with my method. They were taken from Stella's library where Robert Webb provided all models according to the plates in Brückner's 1906 book. Some of them were compounds of two individuals each. Here, for the present paper, one part of the compound was eliminated. These are indicated by the term "half". Except one which has 60 faces, all are icositetrahedra. Again except one which has pentagons as faces, all consist of hexagons.

## Discussion

In my opinion, noble polyhedra are objects of great beauty. This is particularly obvious in the elegant shapes with regular faces. On the other hand, the complex representatives are particularly appealing, because they manage to assemble confusing obliquely angled faces into beautiful, highly symmetrical structures. It is, of course, best to have such a polyhedron in front of you as a physical object. Brückner

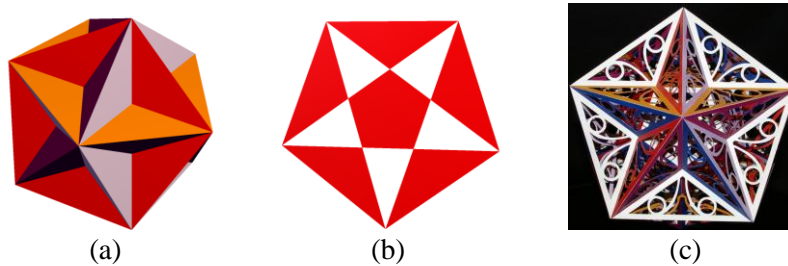


must have felt the same way, because he made 110 paper models for his 1906-book. He did it without Stella4D!

Apart from the Platonic and the Kepler-Poinsot bodies, I have made 3 paper models of noble polyhedra to date. Pictures of them can be seen in Figures 4, 8 and 11. Here, as with all my models, it is particularly important to me to make the surfaces transparent so that their position within the body remains visible. I hope that I'll be able to create some of the more complex ones in my style in future years. This is a rather lengthy process: although Stella4D can print nets of the polyhedra I want to build as paper models I have to redesign the nets completely with Corel Draw because of the patterns I want to cut out of the paper. Furthermore I cannot use coloured paper because the tabs of the facelets must have the colour of the neighboured facelet they are glued on. So I print the nets on white paper and cut them out by hand with scissors and knives. So in the case of W110\_01 (Figure 9 (a)), 2940 facelets must be joined together which takes a lot of time.

All in all, my list of noble polyhedra contains 61 shapes. In addition there are two infinite families, namely the disphenoids and the crown polyhedra. I don't think that this list is complete for several reasons:

- Creating noble polyhedra in this way supports many identical shapes from different source polyhedra. In omitting the repetitions, I may have inadvertently omitted a new finding.
- All efforts to produce isohedral facetings of the truncated icosidodecahedron (W16) were unsuccessful.
- Furthermore there might be other isogonal polyhedra or polyhedra with a subset of isogonal vertices like the duals of the uniform polyhedra that can be faceted to new noble ones.
- And finally Stella is struggling a little with facets which visit the same vertex twice. So I couldn't realize those constructions of wreath polyhedra and V-faced polyhedra described by Branko Grünbaum [3, 4]. Figure 11 shows such a noble polyhedron made of 12 decagrams where each corner of the faces is visited twice. I couldn't realize it in Stella.




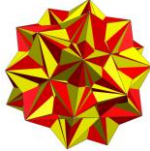


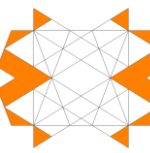
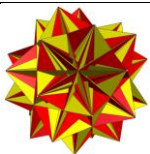
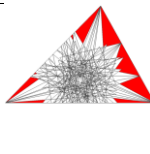
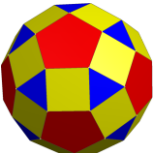
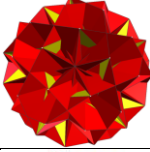
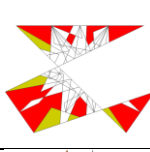
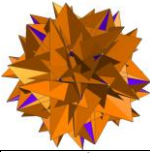
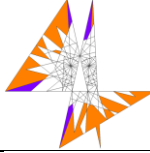
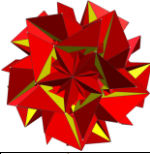

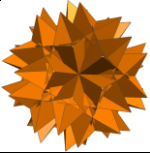
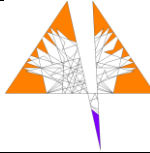
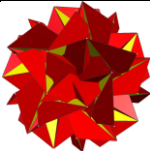
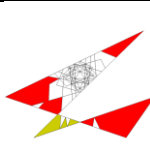
**Figure 11:** (a) *deca\_dod*, (b) *face of deca\_dod*, (c) *paper model of deca\_dod*.

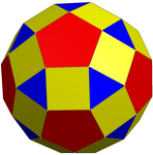
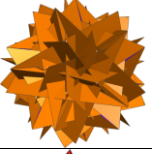
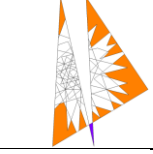
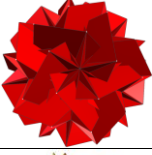
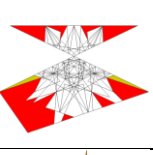
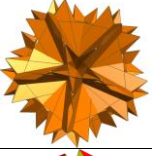
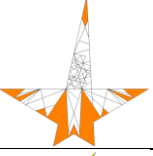
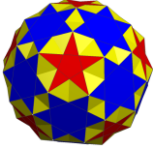
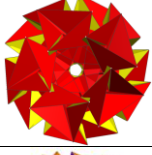
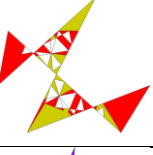
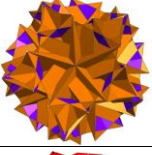
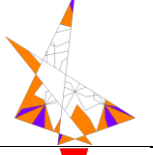
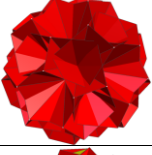
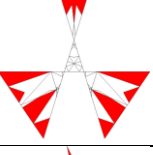
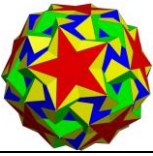
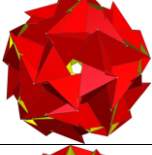
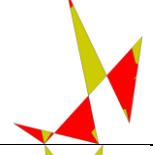
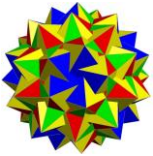
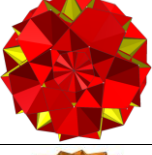
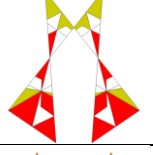
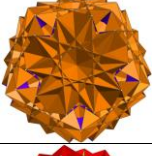
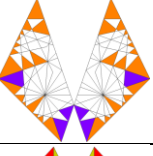

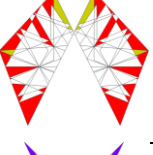
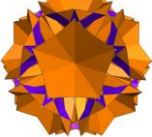
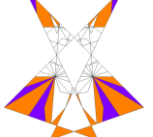
### Acknowledgements

As far as I know no systematic research for noble polyhedra has been carried out since the days of Max Brückner. So perhaps the time has come now for a mathematical expert to do so.

The author thanks Robert Webb for his valuable support in the present paper!

**Table 1:** Noble polyhedra (abridged version: only previously unpublished examples are listed). The complete table is too extensive to be printed here. It can be downloaded from <http://www.polyedergarten.de/TableNoblePolyhedra.pdf>.

Source Polyhedron Wenninger Number	Source Polyhedron Wythoff Symbol	Source Polyhedron	Noble Polyhedron Number	Dual of Noble Polyhedron	Noble Polyhedron	Face	Number of Faces	Face Type
<b>Nonconvex noble polyhedra with non-regular faces previously unpublished</b>								
W12	2   3 5		W12_02				120	3-gon
				W12_02_d			30	12-gon
			W12_06				120	3-gon
W14	3 5   2		W14_03				60	6-gon
				W14_03_d			60	6-gon
			W14_05				60	5-gon
				W14_05_d			60	5-gon
			W14_06				60	5-gon

W14	3 5   2		W14_06_d			60	5-gon
			W14_07			60	5-gon
			W14_07_d			60	5-gon
W110	5/2 3 3		W110_01			60	6-gon
			W110_01_d			60	6-gon
			W110_02_sd			60	6-gon
W112	5/3 3 5		W112_01_sd			60	6-gon
W115	5/3 5/2 3		W115_01			60	6-gon
			W115_01_d			60	6-gon
			W115_02			60	6-gon
			W115_02_d			60	6-gon

### References

- [1] M. Brückner. *Über die Gleicheckig-Gleichflächigen Diskontinuierlichen und Nichtkonvexen Polyeder*. Halle, 1906.
- [2] M. Brückner, R. Stratton, U. Mikloweit. *Concerning the Isogonal-Isohedral, Discontinuous and Nonconvex Polyhedra*. Colorado Springs, 2009. ISBN 978-0-692-00323-7.
- [3] B. Grünbaum. “Are Your Polyhedra the Same as My Polyhedra?” *Discrete and Computational Geometry: The Goodman-Pollack Festschrift*. B. Aronov, S. Basu, J. Pach, and Sharir, M., eds. Springer, New York, 2003, pp. 461–488.
- [4] B. Grünbaum. “Polyhedra With Hollow Faces.”, *POLYTOPES: Abstract, Convex and Computational*”, *Proc. NATO - ASI Conference*. Toronto 1993, T. Bisztriczky, P. McMullen, R. Schneider and A. Ivic' Weiss, eds., Kluwer Acad. Publ., Dordrecht, 1994, pp. 43–70.
- [5] E. Hess. *Über die Zuegleich Gleicheckigen und Gleichflächigen Polyeder*. Kassel, 1876.
- [6] J. F. C. Hessel. *Übersicht der Gleicheckigen Polyeder und Hinweisung auf die Beziehungen Dieser Körper zu den Gleichflächigen Polyedern*. Marburg, 1871.
- [7] R. Webb. *Stella: Polyhedron Navigator*. <https://www.software3d.com/Stella.php>.
- [8] R. Webb. <https://www.software3d.com/NobleSnub.php>.
- [9] M. Wenninger. *Polyhedron Models*. Cambridge University Press, 1971. ISBN 978-0-511-56974-6.