Wallpaper Patterns for Lattice Designs

Laura Taalman[1] and Carolyn Yackel[2]

[1]James Madison University; taalmala@jmu.edu
[2]Mercer University; yackel_ca@mercer.edu

Abstract

Mathematicians have established that there are exactly seventeen plane symmetry groups, known as "wallpaper patterns." Multiple mathematical fiber arts papers describe subsets of those patterns that are suitable for textile design with grid-based media such as knitting and cross-stitch. This paper extends the existing library of these patterns to include orientation-based variations and four-fold rotations. In particular we enumerate all plane lattice symmetries that can be generated from square tiles using lattice translations and combinations of other lattice symmetries. This work has application beyond mathematics as we also provide a useful patterning tool for knitting and fiber arts designers; we introduce Symmetry Generator software created using Python/Processing and OpenSCAD which provides a method for exporting plane lattice symmetry designs as hand knitting patterns or as physical punch cards for use in knitting machines.

Motivation

The first author owns a punch card driven knitting machine. As a computational designer, she wanted to create a tool that would allow for the input of a square of black and white pixels, from which the wallpaper patterns could be generated as punch cards for the knitting machine. As mathematicians we wanted to visually generate *all* possible symmetry-based punch card patterns so that a knitwear designer could select the most appealing patterns to make into punch cards.

To solve this problem, we used a discrete square lattice when programming in Python/Processing to create the multi-functional Symmetry Generator [7], which will prove useful to educators, students, and fans of symmetry and design as well as designers of knitwear, cross-stitch. The Symmetry Generator is an interactive digital interface that uses only square tile images and unit translations to make every potential wallpaper symmetry subject to redundancy and programming constraints. We show that seven of these symmetries have multiple design variants. A great number of modes and toggles in the Symmetry Generator empower the user to further investigate symmetries at a deep level, and make the software useful as a pedagogical and research tool.

The second section of this paper describes the limitations on the possible wallpaper designs that arise from our use of a square tile in the Symmetry Generator. The third section includes a theorem enumerating all wallpaper designs available using a square tile with a lattice design and certain translation constraints. The concluding section summarizes these results and points to future work.

Allowable Symmetries for Square Tile Wallpaper Designs

Amazingly, there are just seventeen plane symmetry groups generated by translations and reflections of the plane; see [3, 1, 4]. Examples of these groups based on leaf motifs are shown in Figure 1. Throughout this paper we will refer to symmetries by their IUC notation, and also give the corresponding orbifold notation in Figures 1 and 4 [1].

In order to generate planar patterns, our Symmetry Generator starts with a generic square input starting tile, applies the symmetries of the square dihedral group D_4 to populate a two-by-two array of tiles, and then

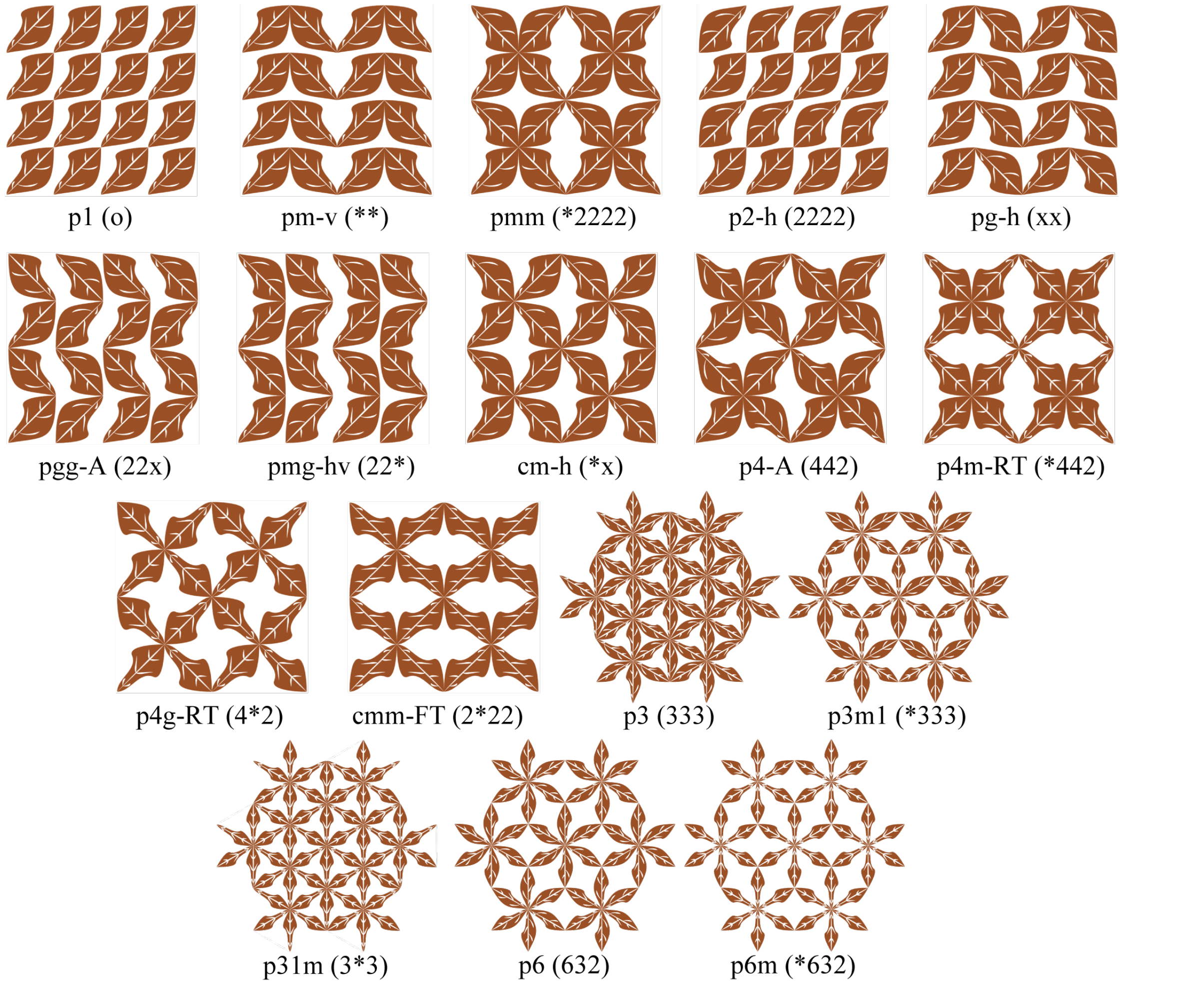


Figure 1: *The seventeen wallpaper patterns using a leaf motif, with IUC and orbifold notation. Modifiers h/v denote horizontal and vertical versions; modifiers A/B denote other variants; modifiers RT/FT denote the need for tiles with internal symmetry.*

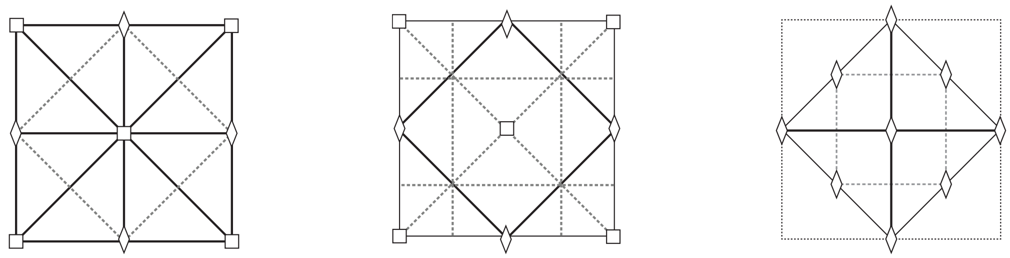
tiles the plane by translating by multiples of two tile lengths horizontally and vertically. The square input tile is the *fundamental domain* of the wallpaper pattern. We define the *translation unit* of a generated pattern to be the smallest region that could tile the plane by horizontal and vertical translation.

A pattern created with the Symmetry Generator necessarily has a translation unit that spans either 1×1, 1×2, 2×1, or 2×2 tile lengths. With these constraints, and considering the cell structure diagrams in [5], we can create a base set of design-relevant variations that cover 9 of the 17 wallpaper patterns; see the first part of Theorem 1. In future work we may expand our algorithm to consider longer and fractional translations which can produce further design variations.

The reader will note that the last five patterns shown in Figure 1 each has a point of order three or order six symmetry. With the assumption of any rational underlying grid, the set of potential wallpaper symmetries is reduced from seventeen to twelve, because the symmetries requiring a 60° rotation inherently move a grid point by a rational multiple of $\sqrt{3}$, see [6]. This eliminates wallpaper symmetries p3, p3m1, p31m, p6, and p6m from consideration in our Symmetry Generator.

The three wallpaper patterns p4, p4m, and p4g shown in Figure 1 require a 90° rotation to act as an isometry on the plane, meaning that in order to include such designs in the Symmetry Generator we must assume a square starting tile. This explains our decision to input square images into the Symmetry Generator and means that we are choosing to ignore that most knit stitches fail to have the same height as their width. Contrastingly, Goldstine did not ignore this property of knit stitches, which is why her art piece *Double Knitting Groups* necessarily depicted only nine symmetries [2].

We also note that in order to include the wallpaper patterns p4m, p4g, and cmm in Figure 1 using only translations of the starting tile, the Symmetry Generator requires special square starting tiles with certain internal symmetries. Figure 2 shows the cell structure diagrams for these three patterns (reproduced from Schattschneider [5]); note in each case there are lines of symmetry which are at 45° angles from each other.



(a) *Symmetries of p4m* **(b)** *Symmetries of p4g* **(c)** *Symmetries of cmm*

Figure 2: *Examples of square grid units for three wallpaper symmetries. Bold show mirror reflections; grey dotted lines show glide reflections; solid lines show outlines of the translation unit; tiny dots indicate the outline of the centered cell. Diamonds mark two-fold centers of rotation; squares mark four-fold centers of rotation.*

From the cell structure diagrams in Figure 2 we can see that the patterns p4m and p4g can be generated from a 2×2 translation unit if we start with a tile that has internal mirror reflection across a diagonal. The pattern cmm either requires a starting tile with 180° rotational symmetry or a larger translation unit. We are restricting to translation units of at most 2×2 tile lengths in the Symmetry Generator algorithm, so we choose the former. In Figure 3a we show a 2×4 translation unit that generates cmm; we instead use the special starting tile shown in Figure 3b to enable the use of a 2×2 translation unit in the Symmetry Generator.

(a) *A 2×4 translation unit configured in a cmm symmetry pattern.* **(b)** *A tile with 180° rotational symmetry.* **(c)** *A 2×2 translation unit configured in a cmm symmetry pattern.*

Figure 3: *We can use any tile to generate cmm symmetry with a 2×4 primitive cell (a), but if we use a rotationally symmetric tile (b) we can generate cmm symmetry with a 2×2 primitive cell (c).*

Enumerating Wallpaper Patterns for Lattice Designs

For our design considerations we now restrict our attention to lattice-based designs of tiles that contain lattice points, to be thought of as knit stitches, cross-stitch, or other grid-based textile media.

Given a plane subdivided into an integer square lattice of punches/stitches, we can consider what we call the *lattice plane symmetries*: horizontal and vertical unit translations, horizontal and vertical reflections, diagonal reflections, horizontal and vertical unit glides, and rotations of orders two and four that preserve the lattice structure. For the rest of this paper, all symmetries will be assumed to be lattice symmetries.

Following the terminology in the previous section, we will call any square subset of the lattice plane a *tile*. For the purposes of machine knitting, tiles should have side lengths that divide the punch card width or machine repeat, but in general tiles can be of any size. The square condition on tiles enables us to consider order four rotations, but can be relaxed to be rectangular for plane patterns that do not include such rotations.

Our main theorem enumerates all possible types of *plane lattice patterns*, that is, all patterns that can be generated from tiles using lattice translations and combinations of other lattice symmetries. This enumeration is also a count of the types of patterns that can be obtained using the authors' Symmetry Generator tool.

In this paper we will consider only translations and glides of one or two full tile widths; this is sufficient to recover variations of all non-threefold wallpaper symmetries. In future work we may generalize to fractional and longer translation and glide lengths. Given any plane lattice pattern there is thus a 1×1, 1×2, 2×1, or 2×2 translation unit that covers the plane by translations.

Theorem 1. *There are exactly 21 plane lattice patterns for discrete lattice designs that arise from unit translations and symmetries of a square tile; see Figure 4.*

a) Exactly 18 plane lattice patterns can be generated from a starting tile using lattice translations together with combinations of other lattice symmetries. Together these patterns exhibit instances of 9 of the 17 classical wallpaper patterns.

b) We can exhibit instances of 3 additional wallpaper patterns in the lattice plane by considering 2×2 *translation units constructed from starting tiles with internal symmetries.*

Considering the length restriction we have imposed on translation units, which we will use throughout the theorem, there are exactly eleven non-translational symmetries that we can apply to a starting square input tile to create a plane pattern that preserves the underlying lattice: vertical and horizontal edge reflections, vertical and horizontal edge glides, vertical and horizontal mid-tile glides, two types (based on placement) of four-fold rotations at tile corners, two-fold rotations at vertical and horizontal tile edge midpoints, and two-fold rotations at tile corners. In the proof of Theorem 1, these are the symmetries that we will systematically consider, roughly in this order. The proof is illustrated by diagrams in which a thick bold line indicates a mirror line, a thick medium grey dotted line shows a line of glide symmetry, a small dark outlined white square shows a point of four-fold rotation, and a small dark outlined white diamond shows a point of two-fold rotation. On the sides of the grids in the diagrams, lines with vertical bars show translation units.

Proof: To prove part (a) we will generate all possible planar patterns from a starting tile by systematically adding translations and various combinations of other symmetries until the entire plane is determined. Note that because we aim to construct a full enumeration suitable for design purposes, we will distinguish between plane lattice patterns if they look visually different, even if they share the same wallpaper symmetry group.

As a trivial case, if we start from one tile with no internal symmetries and apply horizontal and vertical translations of one tile width then we obtain the planar pattern p1. This pattern and all other named patterns in this proof can be found in Figure 4.

Case 1: Edge reflections

Starting from one tile, suppose we add a vertical edge reflection. This reflection forces the horizontal translation unit to be two tile widths (instead of one), and thus determines one infinite horizontal strip of our planar pattern; see Figure 5a.

If we now add a horizontal edge reflection, we determine the rest of the planar pattern and generate pmm, as shown in Figure 5b. Or if we instead add a mid-tile vertical glide, we generate cm-v; see Figure 5c. In a

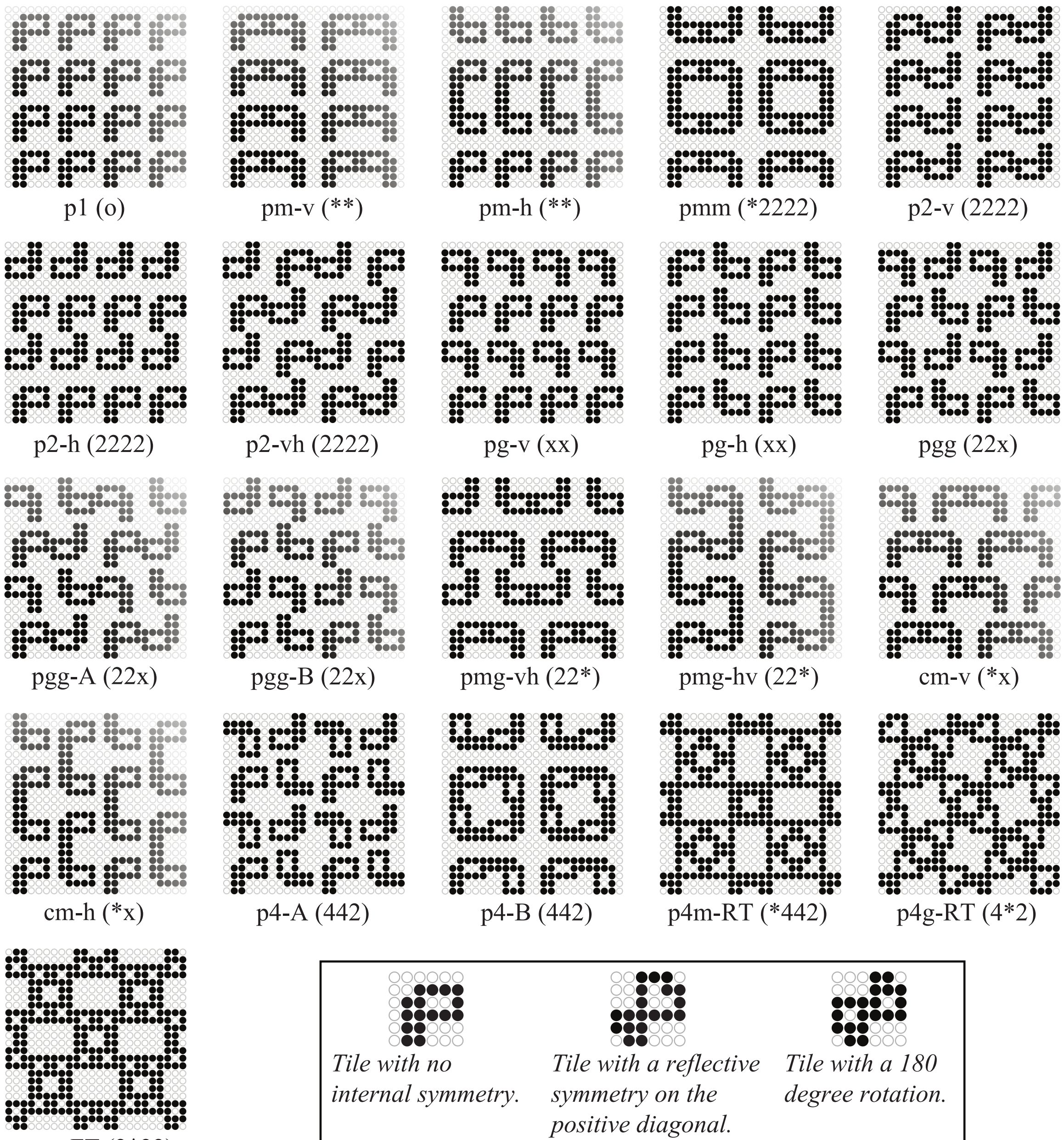
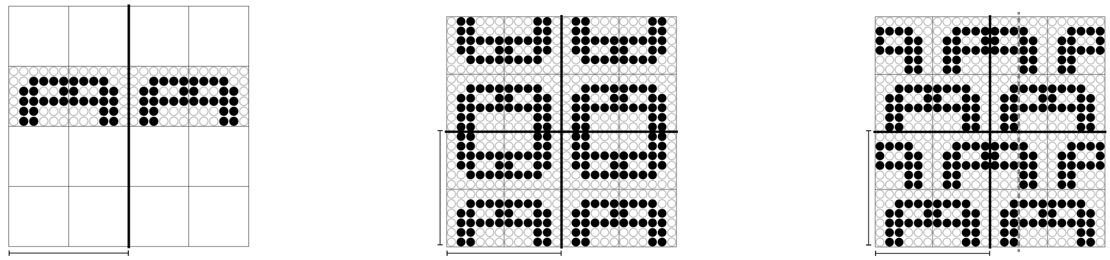


Figure 4: *The 21 possible wallpaper patterns for discrete lattice designs, with IUC and orbifold notation. The first 18 patterns are generated from the left tile in the key, the next two from the reflectively symmetric tile in the key, and the last from the rotationally symmetric tile in the key.Compare with Figure 1.*

similar fashion we could add a vertical translation of one tile height to obtain pm-v or a horizontal mid-tile glide to obtain pmg-vh. It is an easy exercise to check that adding any other glide or rotation to our initial reflection would either be redundant or contradictory.



(a) *One tile and one vertical edge mirror determine a horizontal strip.*

(b) *Adding a perpendicular edge mirror determines the pattern pmm.*

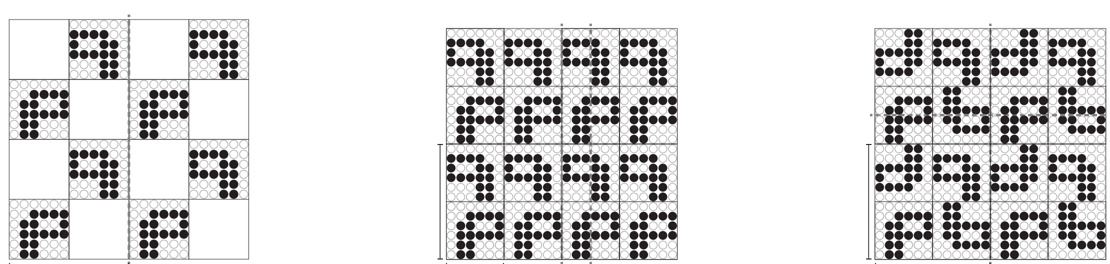
(c) *Adding a parallel mid-tile glide instead determines the pattern cm-v.*

Figure 5: *Plane patterns generated from a vertical edge reflection and other symmetries.*

If we instead start with a horizontal edge reflection, then similar arguments can be used to obtain the additional planar patterns cm-h, pm-h, and pmg-hv. Note that since the starting tile is not assumed to have any internal symmetries we cannot in general have mid-tile reflection axes, and thus we have now exhausted all cases that include reflections.

Case 2: Edge glides, no edge reflections

We can now assume that no reflections are present; suppose that we start with a vertical edge glide whose translation component is one tile height. Because we are restricting to one and two tile width translations, this determines half of the plane in a checkerboard pattern; see Figure 6a. If we add a mid-tile vertical glide then we obtain the planar pattern pg-v; see Figure 6b. Adding a mid-tile horizontal glide instead generates pgg-B, as shown in Figure 6c. All other non-reflection symmetries that we could add are either contradictory, induce reflections, or have arisen in Case 1. In a similar fashion, by starting with a horizontal unit edge glide we can obtain the patterns pg-h and pgg-A.



(a) *One tile and one vertical edge glide determine an infinite checkerboard region.*

(b) *Adding a mid-tile vertical glide determines the pattern pg-v.*

(c) *Adding a mid-tile horizontal glide instead determines the pattern pgg-B.*

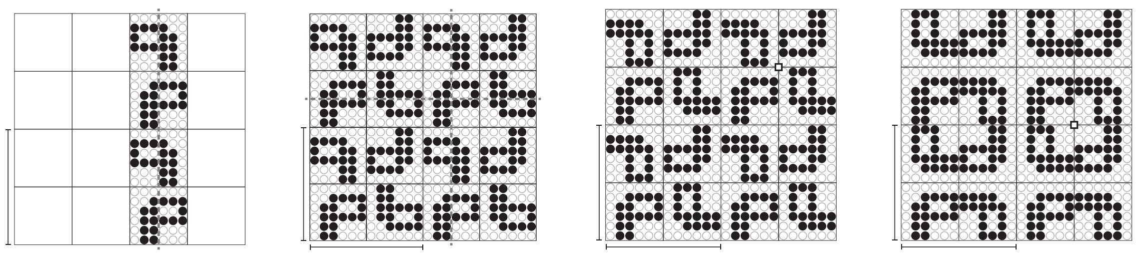
Figure 6: *Plane patterns generated from a vertical edge glide and other (non-reflective) symmetries.*

Case 3: Mid-tile glides, no edge reflections or edge glides

Suppose we now start with a mid-tile glide whose translation component is one tile height. This determines an infinite vertical strip of the pattern, as shown in Figure 7a. Adding a horizontal mid-tile glide generates the pattern pgg; see Figure 7b. An easy check shows that no other new symmetries can be added to the original mid-tile glide to obtain a planar pattern not already considered in Case 1 or Case 2.

Case 4: Rotations only

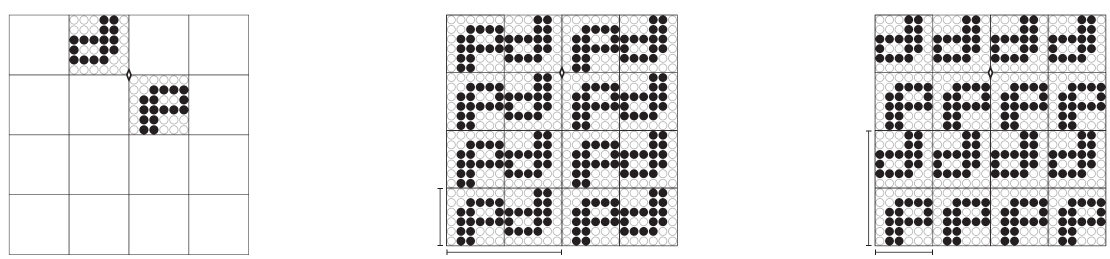
It now remains to consider only patterns generated from combinations of 2-fold and 4-fold rotations. Suppose we begin with a 4-fold rotation at a tile corner. If this corner is at the top right or bottom left of our original tile then we immediately obtain the planar pattern p4-A; see Figure 7c. If the 4-fold rotational corner is instead placed at the top left or bottom right of our original tile then we obtain p4-B; see Figure 7d. (The "original tile" is considered to the the one with the "P" design in both cases.)



(a) *One tile and one vertical mid-tile glide determine an infinite vertical strip.* **(b)** *Adding a mid-tile horizontal glide determines the pattern pgg.* **(c)** *A 4-fold rotation placed at the top right corner of the "P" tile induces the p4-A pattern.* **(d)** *A 4-fold rotation placed at the bottom right corner of the "P" tile induces the p4-B pattern.*

Figure 7: *Left two plane patterns generated from a vertical mid-tile glide and other (non-reflective, non-edge-glide) symmetries. Right two plane patterns generated from 4-fold corner symmetries.*

Now assume there are only 2-fold rotations, starting with one at any corner of our tile. This determines only one other tile, on the other side of that corner; see Figure 8a. If we choose to apply a vertical translation of one tile height at this point we obtain the plane pattern p2-v, as shown in Figure 8b. If we instead apply a horizontal translation of one tile width we obtain p2-h, as shown in Figure 8c.

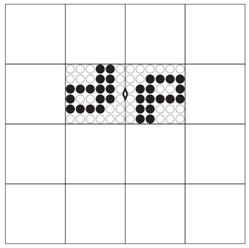


(a) *A 2-fold rotation at the top left corner of the original "P" tile determines one more tile.* **(b)** *Limiting the translation unit to* 1×2 *determines the plane pattern p2-v.* **(c)** *Limiting the translation unit to* 2×1 *determines the plane pattern p2-h.*

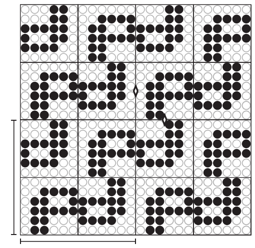
Figure 8: *Plane patterns generated from two-fold corner symmetries.*

Finally, suppose we start with a 2-fold rotation at the midpoint of a vertical tile edge. This determines an infinite horizontal strip of the pattern as shown in Figure 9a. If we add a 2-fold rotation at the midpoint of the horizontal tile edge we obtain p2-vh, as shown in Figure 9b. Adding any other lattice symmetries, including tile-width-one translations, would either repeat previous plane patterns or be inconsistent.

To prove part (b), we need only exhibit one instance of each of the remaining non-threefold wallpaper symmetries: p4m, p4g, and cmm, as we have done in Figure 4 by using the special tiles in the key. We will not attempt to find all instances or to classify other symmetries using these special tiles in this paper. To realize these three symmetries within the square lattice plane and using only a 2×2 translation unit, we have to consider starting tiles that have internal reflective diagonal symmetry or internal 2-fold rotational symmetry.



(a) *A 2-fold rotation at the left edge midpoint of the original "P" tile determines one more tile.*



(b) *Adding a 2-fold rotation at the bottom edge midpoint defines the plane pattern p2-vh*

Figure 9: *Plane patterns generated from two-fold edge midpoint symmetries.*

In Figure 4 the p4m symmetry was obtained by starting with a reflective-diagonal tile and applying the pattern pmm to that tile. When combined with the internal symmetries of the special reflective tile, we obtain all of the symmetries in p4m as described in [5]. Similarly, the p4g symmetry can be obtained by applying the pattern pgg to the same tile, and cmm can be obtained by applying the pattern pmm to a 2-fold rotational starting tile. In future work we will explore complete enumerations of such pattern types. □

Conclusion and Future Work

In this paper we have provided a complete set of design variations for wallpaper patterns on discrete lattice designs constrained by translation unit size. In future work, we intend to write about other features of the Symmetry Generator, including its exceptional capability to generate huge collections of design variants for the types of two-color designs enumerated in [1, 4]. We will also explore further design restrictions imposed by punch card knitting machine repeats. Further, we hope to extend the functionality of the Symmetry Generator to fractional and expanded translation units.

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