Cardboard Construction of the Sphere by the Stereographic Projection

María García Monera

Universitat de València, Spain; monera2@uv.es

Abstract

Based on the previous work by Felix Klein, Alexander von Brill, and John Sharp, we give a new method to build a cardboard solid sphere. The templates have been obtained by the inverse of the stereographic projection of two pencils of lines. This gives a technique of construction by rotating a plane around an axis crossing the surface.

Introduction

When Felix Klein (1849-1925), an outstanding mathematician, was preparing to teach geometry, he strove to aid his mental models of surfaces by creating physical models. To achieve this goal, he and Alexander von Brill (1842-1935) set up a workshop at the University of Munich to design a wide variety of geometric surfaces with their students. Through this endeavor, more than 400 models were designed in plaster, wire, and paper, using designs based on the geometric properties of the surface. The Martin Schilling company sold the models to universities and museums around the world. For various reasons, the company closed and today museums, like the Science Museum of London, exhibit some of these models.

At the end of the last century John Sharp (see [4, 5, 6]) explained a general method to design a wide variety of cardboard surfaces using the same technique: *Sliceforms*. This method starts by intersecting a surface with two families of parallel planes. Pieces of paper cut in shapes of these intersections, when scored correctly, can be assembled to make a model. These pieces are called *templates*. The technique has only one premise, a plane is needed through each maximum and minimum of the function defining the surface. Using

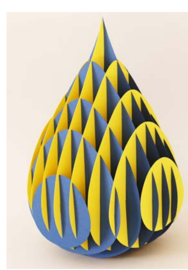


Figure 1: *A drop water designed using Sliceforms.*

these techniques we designed some geometrical models with Mathematica© (see [1, 2]). A video tutorial on how to assemble the torus presented in [1] can be found at https://youtu.be/rft4EJkVZ9Y. In this model, the templates are the Villarceau sections.

In this paper, we propose a technique of construction where two families of parallel planes are not needed. In the first part, we present a construction of a cardboard solid sphere by the inverse of the stereographic projection. Note that all cross sections of the sphere are circles. Hence, if we want to build a physical model of the sphere, we need planar curves that map to circles on the sphere via the stereographic projection. These planar curves are either lines or circles. In this note, we shall consider some specific lines contained on a plane whose projection on the sphere will be used as templates to reproduce a real model.

In mathematics, a pencil is a family of geometric objects with a common property, for example the set of lines that pass through a given point or a set of planes passing through a straight line. Our designs are based on the position of some representative lines of a pencil of lines. However, as in John Sharp's models, the complexity of the model increases with the number of templates, in our case, with the number of pencils and lines. For this reason, only two pencils have been considered in the design of the models. This technique only permits at most four entire templates for each pencil. If we add more templates, then some of them have to be split in half. For this reason, only four lines has been considered for each pencil. Finally, just for aesthetic purposes, the angle between the selected lines in a pencil is always $\frac{k\Pi}{4}$, $k \in \mathbb{Z}$ and there is at least one line in common with both pencils. Depending on the position of the lines with respect to the sphere we finally present four different models.

In the second part of the paper, we explain a technique of construction based on the design of these models. This technique consists of the rotation of a plane around two axes crossing the surface.

The Stereographic Projection

The stereographic projection is a bijective function that projects a sphere onto a plane, this projection is defined on the entire sphere except at the projection point. Conversely, we can project a plane onto a sphere by the inverse of the function.

Let $S^2 \subset \mathbb{R}^3$ be the unit sphere centered at the origin $(0,0,0) \in \mathbb{R}^3$. Taking the North Pole, N, as the projection point, the stereographic projection from this point to the plane $z = -1$ is given by:

$$\begin{array}{rccc} \Omega: & S^2 & \longrightarrow & \mathbb{R}^3 \\ & (x,y,z) & \mapsto & \left(\frac{2x}{1-z}, \frac{2y}{1-z}, -1\right). \end{array}$$

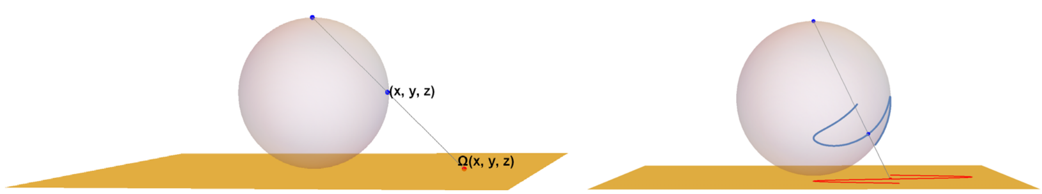


Figure 2: *(a) Image of a point (x, y, z) by Ω, (b) image of a curve (colored red) by Ω (colored blue).*

Conversely, the inversion of the stereographic projection is given by:

$$\begin{array}{rccc} \Omega^{-1}: & \mathbb{R}^3 & \longrightarrow & S^2 \\ & (x,y,z) & \mapsto & \frac{1}{x^2+y^2+4}(4x, 4y, x^2+y^2-4). \end{array}$$

Construction of the Sphere

Consider certain selected lines in a pencil, Γ_1, which intersect into a point P_1 and are contained on the plane $z = -1$. Its image by Ω^{-1} is a family of circles on S^2 which intersect at the points N and $\Omega^{-1}(P_1)$. Since we need two families of templates to build the model, we shall consider another pencil of lines, Γ_2, contained on the same plane and intersecting into another point P_2.

Depending on the position of P_1 and P_2, we have considered four different cases. We shall start with the easiest case of one pencil based at the South Pole and continue on to pairs of pencils. The templates are computed as the intersection between the sphere and the plane containing each circle. Finally, the slots are given by the intersection line between the templates.

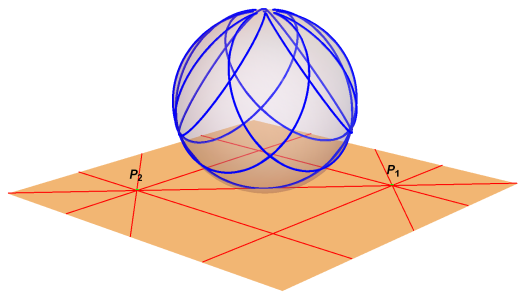
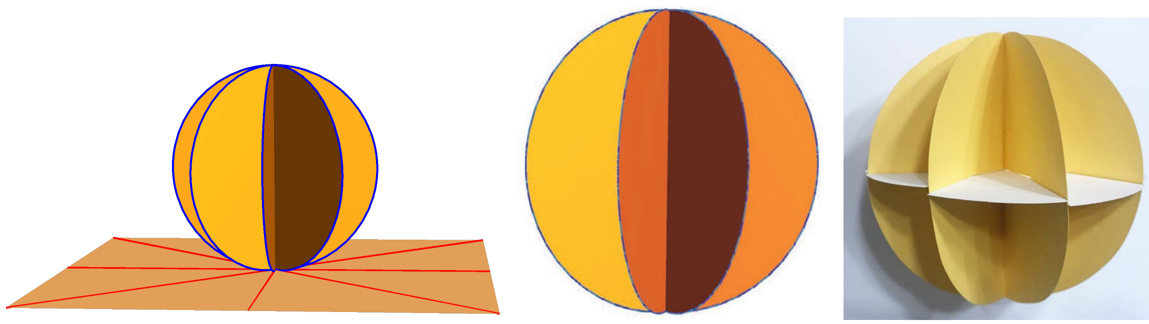


Figure 3: *The image by Ω^{-1} of some lines in pencils Γ_1 and Γ_2.*

Case 1

Let $P_1 = P_2 = S$, where S is the South Pole. In this case, both families of lines are coincident and we obtain only one family of templates.



(a) **(b)** **(c)**

Figure 4: *(a) Coincident pencils of lines for $P_1 = P_2 = S$, (b) virtual model, (c) real model.*

In this case, it is not possible to maintain the angle between the templates made with cardboard or paper, because we have only one family of templates. This does not happen in the following models, where we obtain two different families. This model of the sphere requires an additional template. The next figure shows the templates of the model, where the additional template has been located on the left hand side.

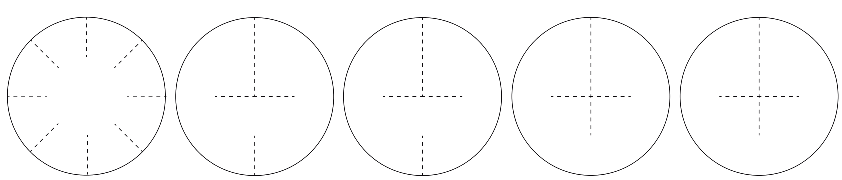
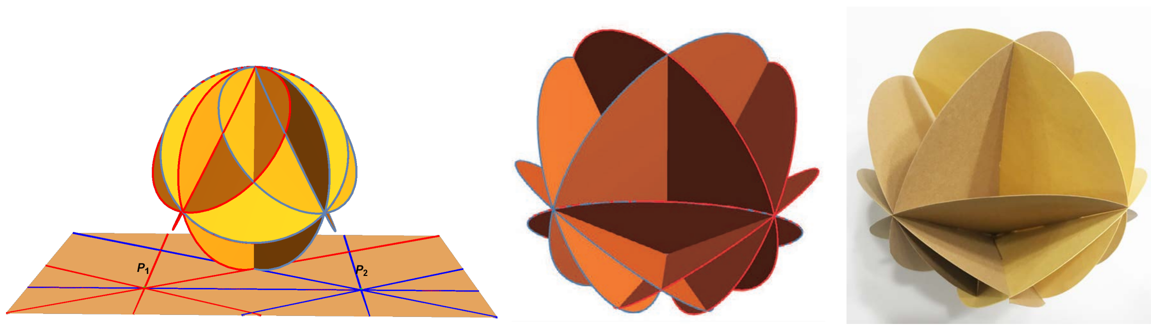


Figure 5: *Templates of the model $P_1 = P_2 = S$.*

Case 2

Consider now that P_2 is given by the rotation of P_1 an angle of $\frac{\pi}{2}$ around the z-axis. In this case, Γ_1 and Γ_2 intersect in a line through the points P_1 and P_2. This means that there is one circle, and therefore one circular template, that belongs to both families.

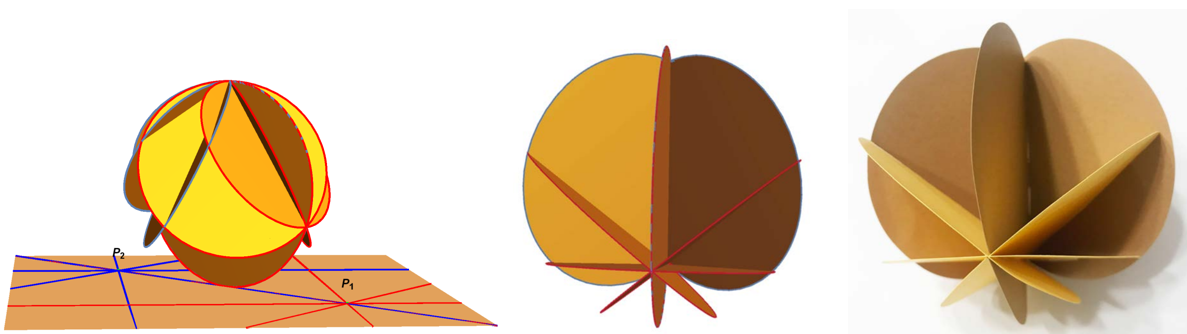


(a) **(b)** **(c)**

Figure 6: *(a)* Γ_1 *and* Γ_2, *with* P_2 *obtained by rotating* P_1 *an angle of* $\frac{\pi}{2}$, *(b) virtual model, (c) real model.*

Case 3

Consider that P_2 is given by the rotation of P_1 an angle of π around the z-axis. Both pencils of lines intersect in a line through the points P_1 and P_2. In this case, its image by Ω^{-1} is a great circle on S^2.

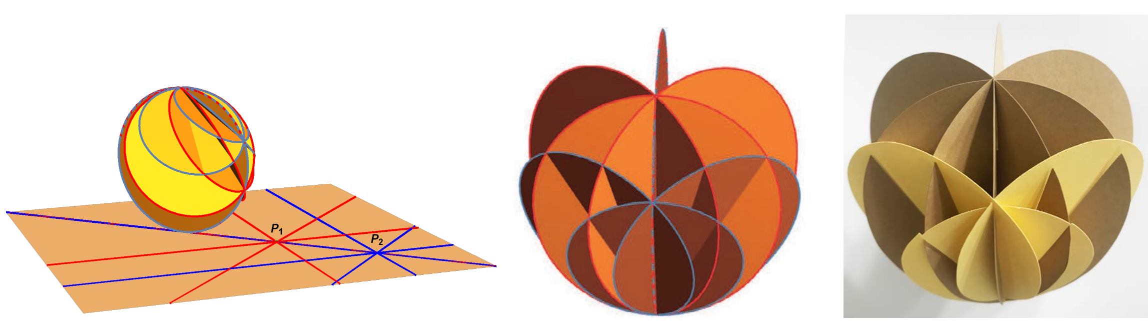


(a) **(b)** **(c)**

Figure 7: *(a)* Γ_1 *and* Γ_2, *with* P_2 *obtained by rotating* P_1 *an angle of* π, *(b) virtual model, (c) real model.*

Case 4

Finally, suppose that P_1, P_2 and the South Pole, S, are contained on the same line. In addition to this, consider that $P_1 \in \overline{SP_2}$, where $\overline{SP_2}$ is the segment defined by S and P_2. The case $P_2 \in \overline{SP_1}$ is the same. Again, the two pencils intersect in a line through the points P_1 and P_2 which gives a great circle on S^2 by Ω^{-1}.



(a) **(b)** **(c)**

Figure 8: *(a)* Γ_1 *and* Γ_2, *with* P_1, P_2, *and* S *aligned, (b) virtual model, (c) real model.*

General Technique of Construction of Cardboard Models

As we stated in the introduction, at the end of the last century, John Sharp explained in [4] a general method to build cardboard models by intersecting the surface with two families of parallel planes, Sliceforms. More recently, in [1, 2], the author presents some models of solids of revolution where the surface has been intersected by two planes rotating around the axis of revolution. The models presented in [3] have been designed by intersecting the surface with two pencils of planes, where the straight line defined by each pencil does not cross the surface. The design of a solid sphere by the stereographic projection that we present in this note generalizes this last technique. Notice that all the cardboard models could have been designed by the intersection of the sphere with two pencils of planes where, unlike in the previous cases, the straight line intersects with the sphere.

Let us see the design of a solid cylinder using this technique. Let C be the cylinder given by $x^2 + y^2 = 1$, $x,\ y \in \mathbb{R}$. Consider now two pencils of planes defined by the x, y-axes. In order to obtain more templates, in this case we rotate the plane around the axis an angle of $\frac{k\pi}{6}$, $k \in \mathbb{Z}$. The outline of the templates, which have been obtained by the intersection of the planes with C, are either circumferences, ellipses or rectangles.

(a) **(b)** **(c)**

Figure 9: *(a) Pencil of planes intersecting with the cylinder, (b) virtual model, (c) real model.*

Finally, let us see the templates of the model. Notice that, as we stated in the introduction, since we are using more than 4 planes for each pencil, some templates have to be split in half.

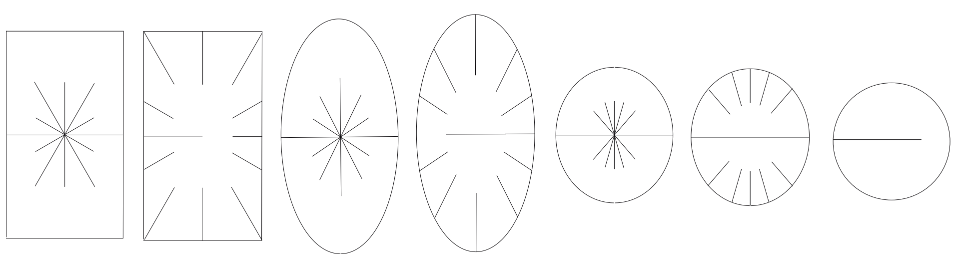


Figure 10: *Templates of the cylinder.*

The advantage of this technique is that all models are built in the same way. Therefore, if you can build a first model, this will help you with the rest of the surfaces. This does not happen with the models of the solid sphere by the stereographic projection, which have some complexity due to the particular design of each model.

Conclusion

This work has been based on the projection on the sphere of some representative lines in two pencils of lines. In future research other models of cardboard solid spheres can be designed with more pencils of lines, as well as by using pencils of circles. Also, a combination of lines and circles can be considered for future designs.

The templates of the models can be downloaded at http://www.uv.es/monera2. A video tutorial showing how to build the spheres can be found at https://youtu.be/Tu1Kxrxg7po.

Acknowledgements

During the review of this article, I sadly learnt that John Sharp has recently passed away. His work was always very inspiring to me, and I would like to thank him for the immense contribution he has made to recreational mathematics. Also, I would like to thank the editor and referees for their useful comments, which have helped me to improve this article in a substantial way.

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References

[1] M. G. Monera and J. Monterde. “Building a torus with Villarceau sections.” *Journal of Geometry and Graphics*, vol. 15, no. 1, 2011, pp. 41–47.

[2] M. G. Monera. “Slicing surfaces with Mathematica.” *Math Art Summit Proceedings*, Brussels, Belgium, May 28–30, 2012. http://etopia.sintlucas.be/3.14/Wiskunst/Proceedings%20MathArtSummitPart1.pdf.

[3] M. G. Monera. “Superficies seccionadas.” *Proceedings of XIII Jornadas de educación matemática de la Comunidad Valenciana*, Alicante, Spain, 2019, pp. 92–102.

[4] J. Sharp. *Sliceforms: Mathematical models from paper sections*, Tarquin publications, 1995.

[5] J. Sharp. “Sliceform Sculptures-a Bridge between Art and Mathematics.” *Bridges Conference Proceedings*,Winfield, USA, July 28–30, 1998, pp. 275–276. http://archive.bridgesmathart.org/1998/bridges1998-275.pdf.

[6] J. Sharp. *Surfaces: explorations with sliceforms*, Tarquin publications, 2004.