Knotty Knits are Tangles on Tori

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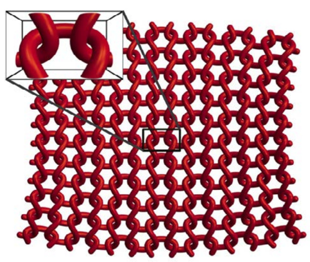
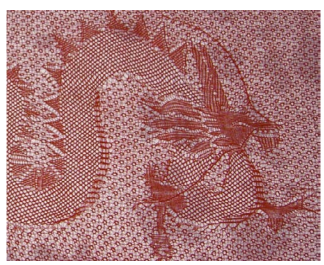
Abstract

In this paper we outline a topological framework for constructing 2-periodic knitted stitches and an algebra for joining stitches together to form more complicated textiles. Our topological framework can be constructed from certain topological "moves" which correspond to "operations" that knitters make when they create a stitch. In knitting, unlike Jacquard weaves, a set of n loops may be combined in topologically nontrivial ways to create n stitches. We define a *swatch* as a mathematical construction that captures the topological manipulations a hand knitter makes. Swatches can capture the topology of all possible 2-periodic knitted motifs: standard patterns such as garter and ribbing, cables in which stitches connect one row of loops to a permutation of those same loops on the next row much like operators of a braid group, and lace or pieces with shaping which use increases and decreases to disrupt the underlying square lattice of stitches.

Introduction



(a) *Schematic of a knitted fabric. It is a periodic structure of slip knots.*



(b) *Textiles with intricate patterns are knit by combining slip knots in specific combinations.*

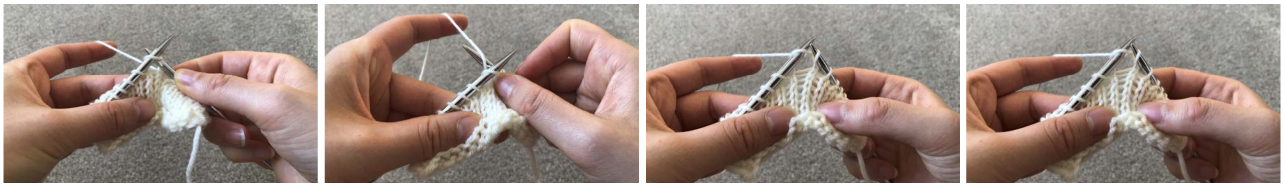
Figure 1

Imagine a one-dimensional curve: entwine it back and forth so that it fills a two-dimensional manifold which covers an arbitrary three-dimensional object – this computationally intensive materials challenge is realized in the ancient technology known as knitting. This process for making functional two-dimensional materials from one-dimensional yarns dates back to prehistory, with the oldest known examples found in Egypt from the 11th century CE [1]. Knitted textiles are ubiquitous as they are easy and affordable to create, lightweight, portable, flexible and stretchy. As with many functional materials, the key to knitting's extraordinary properties lies in its microstructure. The entangled structure of knitted textiles allows them to increase their length by over 100% whilst barely stretching the constituent yarn.

From socks to performance textiles, sportswear to wearable electronics, knits are a ubiquitous part of everyday life. The geometry and topology of the knitted microstructure is responsible for many of these properties, even more so than their constituent fibers. But first, what constitutes a knit?

Knits and purls

Knits are composed of a periodic lattice of interlocking *slip knots*. (Note that these slip knots are not to be confused with the knitting notion of *slip stitches* in which a stitch is moved from the left needle to the right needle without pulling a loop from the active yarn through it.) At the most basic level, there is only one manipulation that constitutes knitting – pulling a loop of yarn through another loop, (see Figure 1a). There are two basic "stitches" produced by this manipulation: a *knit* stitch pulls a loop from the back of the fabric toward the front, whilst a loop pulled from the front of the fabric towards the back is called a *purl* stitch.

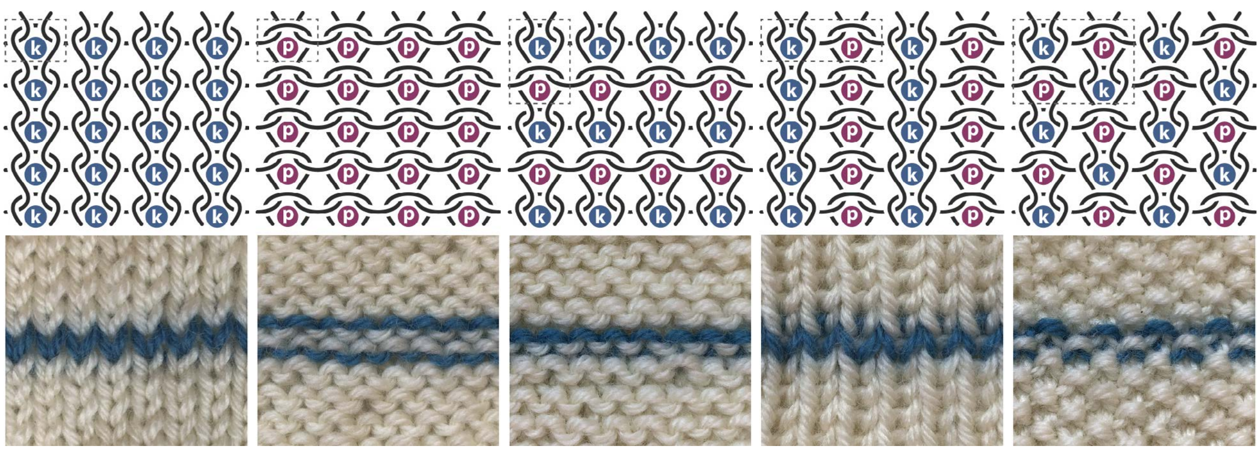


(a) *Knitting begins with loops on two needles. First you insert the right needle tip into the first loop on the left needle.* **(b)** *Then you wrap the free yarn around the right needle clockwise.* **(c)** *The newly formed loop of yarn gets pulled through the loop on the left needle.* **(d)** *Lastly, you slide the loop off of the left needle. It is now captured by the loop you just made and both are caught on the right needle.*

Figure 2: *The process of hand knitting.*

These stitches are actually the same; when viewed from the back, a knit stitch is a purl stitch. Combining these two motifs, there exist thousands of patterns of stitches with immense complexity, each of which has different elastic behavior (see Figure 1b).

A piece of plain-knitted or weft-knit fabric contains only one thread which zigzags back and forth horizontally through the length of the fabric. The process of knitting threads a loop from the active yarn through loops from the previous row. Consecutive knitted stitches are connected to one another horizontally, a direction known as the *course*. Knitted fabric is held together by a square lattice of these slip knots – rows are connected to each other vertically with slip knots. Columns of slip knots form along the vertical direction – called the *wale* – connecting a single thread into a textile.



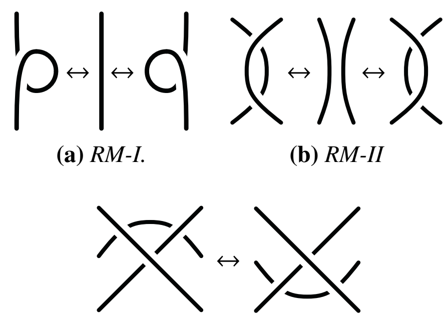
(a) Stockinette *fabric is formed by a lattice of knit stitches.* **(b)** Reverse stockinette *fabric is formed by a lattice of purl stitches.* **(c)** Garter *fabric alternates rows of all knit and all purl stitches.* **(d)** 1×1 ribbing *alternates columns of all knits and all purls.* **(e)** Seed *fabric is a checkerboard lattice of knits and purls.*

Figure 3: *Common fabrics created using knit and purl stitches. Remarkably these fabrics all have very different elastic behaviors, despite being nearly topologically identical. (The exception to this is stockinette and reverse stockinette, which are related by rotational symmetry.)*

Using solely knit and purl stitches, thousands of distinct fabrics can be created, each with different elastic properties. See Figure 3. *Stockinette* fabric is created entirely of knit stitches (Figure 3a). Likewise, *reverse stockinette* is made from entirely purls (or by turning over stockinette fabric) (Figure 3b). Stockinette and reverse stockinette have a preference for negative gaussian curvature. In both fabrics, the bottom and top curl towards the knit side of the fabric, whilst the left and right slides curl towards the purl side. *Garter* fabric alternates rows of knit stitches and purl stitches (Figure 3c). In 1×1 *ribbing*, knits and purl alternate, keeping all stitches in each column the same (Figure 3d). This fabric is very stretchy and has a corrugated appearance. Ribbing fabric is frequently used for cuffs and collars of garments. *Seed* fabric is a checkerboard lattice of knits and purl (Figure 3e). The latter three fabrics lie flat because they have a rotational symmetry in the plane of the fabric that leaves the front and back of these fabrics indistinguishable. Stockinette and reverse stockinette fabrics lack this symmetry and the local deformation of each knit (or purl) stitch is compounded across the entire fabric, with the consequence that it curls. The local topology of stitches, as well as the order in which they appear in the fabric determines the local geometry of the fabric and, therefore, its elastic response.

Knits as knots in $\mathbb{T}^2 \times I$

Topology and entanglement hold textiles together, yet knits are topologically trivial; because a knitted textile is comprised of slip knots, pulling a single loose thread can unravel the entire garment. Knitting is doubly periodic – that is, it lives on a square lattice. Thus, invoking periodic boundary conditions leaves us with a knot that cannot be untangled, see Figure 3,6a.



(a) *RM-I.* **(b)** *RM-II* **(c)** *RM-III*

Figure 4: *These three "moves" on a planar projection of a knot are ambient isotopies that leave the overall topology of the knot unchanged.*

Knot theory provides us with a natural framework to study such entanglement problems. A *knot* is a nontrivial embedding of a circle S^1 into $\mathbb{R}^3$. Likewise, a *link* consists of two or more disjoint circles embedded in $\mathbb{R}^3$. Two knots or links are topologically *equivalent* if one can be transformed into the other via a deformation of the ambient space –known as an *ambient isotopy* – that does not involve cutting the knot or letting the string pass through itself. Knot theory studies topological descriptors of this equivalence, known as *invariants*. It is important to note that while two representations of the same knot must be characterized by the same invariant, two knots that have the same invariant are not necessarily equivalent.

Some branches of knot theory treat the string (the S^1's) as the fundamental object and use the over and under crossings to create an algebraic representation of the knotted object. Here, *planar knot diagrams* – projections of the three-dimensional representation of the knot into a two-dimensional plane keeping the details of the over and under crossings of the knot. In this framework, ambient isotopies take the form of Reidemeister moves, shown in Figure 4 [3]. There are several invariants that are used in this algebraic framework. Some typical examples of this are *linking number* – the number of times a pair of components in a link pass through one another – and *Alexander polynomials* – a Laurent polynomial with integer coefficients which is created by starting with the unknot and using polynomial generators based on changing crossings called *skein relations* to build up to the entire knot of interest [7, 8].

Other branches of knot theory abandon the S^1's in favor of the *knot (or link) complement* which is constructed by placing the knot or link in the 3-sphere S^3 and drilling out a tubular neighborhood around the knot or link. The resulting 3-manifold is the primary object. The 1989 theorem by Gordon and Luecke states that topological invariants of the knot complement are also invariants of the knot itself [4, 5]. However, this is not strictly true for links, especially when looking at fibered manifolds. The following description of knitted knots will blend both of these descriptions, although it may be weighted a little towards the 3-manifold representation. The knot invariants here are topological invariants of the 3-manifold. For instance, the *hyperbolic volume* is the volume of the knot complement with a hyperbolic structure, and the fundamental group of the complement [7].

The ultimate goal of such a topological description is to create an algebra to describe any fabric using textile knots. Such an algebra is *complete* if it can create all fabrics compatible with knitting. The basis set for this algebra are all possible slip knots compatible with knitting. More complex fabrics can be created by "adding" different stitch types together vertically and horizontally. For instance, *garter* is created by "adding" a knit stitch and a purl stitch together vertically, whilst 1×1 *ribbing* is created by "adding" the same two stitches horizontally.

Knits, weaves and other two-periodic textiles live naturally in a space homeomorphic to a thickened torus, $T^2 \times I$. Normally two-dimensional objects with periodic boundary conditions can be represented with a torus. However, knits have over and under crossings, which give this toroidal structure some thickness. This space can be thought of as the glaze on the outside of a donut. Unlike strictly two-dimensional doubly periodic structures, we can't simply identify the left edge with the edge side and the top edge and edge of the knit diagram (see, eg, Figure 3), because the resulting knots are not topologically equivalent [6].

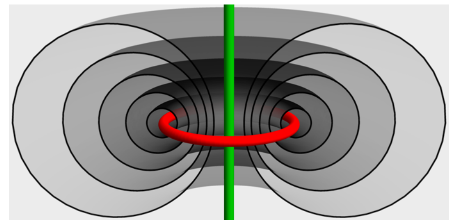


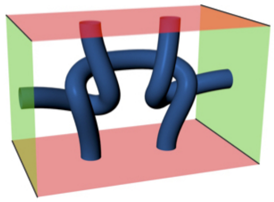
Figure 5: *The thickened torus is isomorphic to* $S^3 - \mathcal{T}_{(Hopf\ link)}$

We turn to 3-manifold topology in order to study these textile knots in their natural space. Any invariant of the manifold created by removing a tubular neighborhood $\mathcal{T}$ from around the knot $\mathcal{K}$ in the 3-sphere, denoted $S^3 - \mathcal{T}_{\mathcal{K}}$, is also an invariant of the knot $\mathcal{K}$. When the knot is not embedded in ambient euclidean space $\mathbb{R}^3$ (as is the case with textile knots living in $T^2 \times I$), we can create the ambient manifold by removing a specific knot or link from S^3. In particular, $T^2 \times I$ is homeomorphic to S^3 minus a *Hopf link*, a pair of embedded circles which pass through each other's centers.

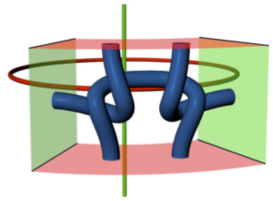
The following is a canonical construction of the 3-manifold complement of our textile knot. We start with a knitted stitch in the thickened torus $T^2 \times I$ (Figure 6a), where the pairs of green and pairs of red sides are identified. In Figure 6b, this is then put into $S^3 - \mathcal{T}_{(\text{Hopf link})}$, where the red and green tubes designate the Hopf link. Note, the green tube connects through infinity. The green sides of the thickened torus in Figure 6c connect by encircling the green circle of the Hopf link. This green cycle is resized to fit in the frame in Figure 6d. The final maneuver to connect up the knitted stitch, in Figure 6e,f, identifies the red faces with one another by wrapping around the red element of the Hopf link.

We now define *standard position* for a link $T^2 \times I$ which has been lifted into $S^3 - \mathcal{T}_{(\text{Hopf link})}$, see Figure 6g. Standard position is a canonical construction of the textile link in S^3. In standard position, the identified sides of the original thickened torus (red and green in Figure 6a) are now annuli. Each annulus has one boundary component isotopic to the component of the Hopf link of the corresponding color. The other boundary is punctured by the other component of the Hopf link. These annuli intersect one another along a curve that connects the two boundary components. The course direction punctures the green surface, and the wale punctures the red surface.

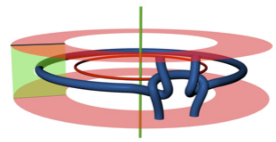
By converting this image into a two dimensional link diagram with planar crossings (Figure 6h). In Figure 6h, there is a dashed rectangle which corresponds to a flattened version of the original knot in $T^2 \times I$. One might ask what conditions exist on knots in the dashed rectangle such that they are knitable? Hand knitters have an implicit notion of what a *stitch* is – a set of manipulations of existing loops and/or free yarn that ends when a loop is passed from the left needle to the right needle. Unfortunately, rigorizing this definition will always require a choice. Is there a level of complexity that is allowed by knot theory, but no human or machine could ever physically create it? Some ambient isotopies of a *bight* – a small continuous segment – of yarn, might be too complex for a knitter to do using only two needles without additional equipment or scaffolding, however topologically, these would always be allowed. For example, twisting a stitch an arbitrarily large number of times or creating an arbitrarily long chain of single crochet are topologically consistent with being knitable.



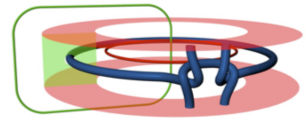
(a) *The knitted stitch lives in the manifold $\mathbb{T}^2 \times I$. Here, green sides are identified and red sides are identified.*



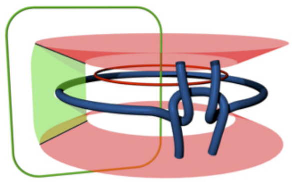
(b) *In order to see the knit stitch as a link in S^3, we created $\mathbb{T}^2 \times I$ by subtracting the tubular neighborhood of a Hopf link from.*



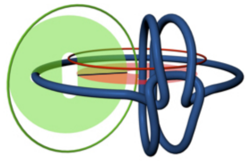
(c) *When the green faces are identified, the knit stitch must link with the green component of the Hopf link.*



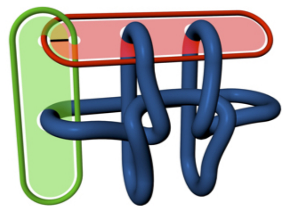
(d) *The green component of the Hopf link is truly an S^1 embedded in S^3.*



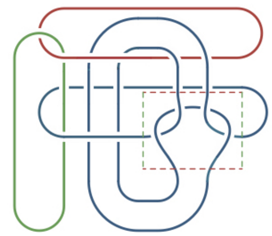
(e) *When the red faces are identified, the knit stitch must link with the red component of the Hopf link.*



(f) *The green and red surfaces in $T^2 \times I$ are a pair of annuli that intersect along a single line.*



(g) *The 2-periodic knit stitch is now a three component link in S^3.*

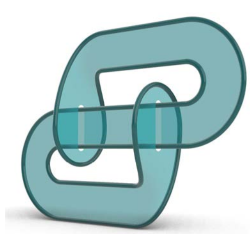


(h) *This planar projection shows the 2-periodic knit stitch in* standard position.

Figure 6: *2-periodic knit stitches naturally live in $T^2 \times I$. However, we can construct 2-periodic knit stitches as three component links in S^3.*

For a knot to be knitable, it must be created from *slip knots*, which are a class of ambient isotopies of a portion of the interval with end points fixed created by pulling bights of that line through one another. From a knitter's perspective, these knots can be unravelled when the both ends of the unit line are pulled on. In $\mathbb{T}^2 \times I$, this class of knots has nontrivial homology around the longitude and trivial homology around the meridian. Each knitted component of the link wraps around the longitude once (horizontal in the figures). Although there is entanglement that keeps the links from coming undone along the meridian (vertical in the figures), the knitted link components do not wrap around the meridian. Instead, they double back on themselves. This implies that in a knitted textile, each row of stitches is connected together along one piece of yarn while neighboring rows are pairwise trivially linked. This is apparent in standard position. The knitted component of the link (blue) is pairwise linked with the green component of the Hopf link (the longitude) and is *trivially linked* – meaning a pair of link components has linking number zero even though it might not be possible to disentangle them – with the red component (the meridian), as shown in Figure 6h.

Since we can now construct the 3-manifold complement of our textile knot, by removing both the Hopf link and our textile link from S^3, we can use many of the conventional tools from knot theory to study the topological invariants of this construction. For instance, the knot theory and 3-manifold topology software SnapPy [2] uses a graphical link editor to construct the 3-manifold complement of a knot or link in S^3 from the planar knot diagram, eg Figure 6h.



(a) *Ribbon knot.*

(b) *Cow-hitch schematic*

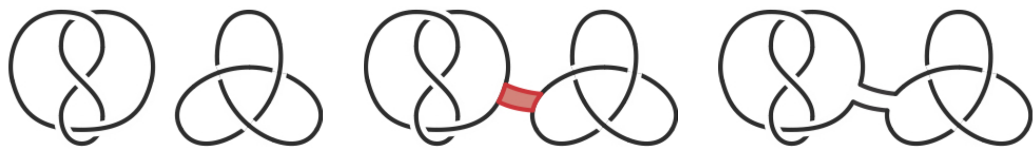
(c) *Cow-hitch knitted*

Figure 7

An examination of many commonly used knitable stitches reveals that all share the property that they are *ribbon*. *Ribbon knots* are knots that bound a self-intersecting disk where all self intersections are *ribbon singularities* – places where the ribbon self intersects form curves that exist only in the interior of the spanning disk [7]. Intuitively, this is not surprising, since all knits are formed by sliding bights of yarn through each other. We conjecture that all knits are ribbon. We will show later that being ribbon is a necessary, but not sufficient, condition for a knot to be knitable.

What types of ribbon knots can be turned into knitable stitches? A class of potentially knitable ribbon knots come from tying other knots or links in the bight and the knitting that into the next row. One example of such a stitch we call the *cow-hitch* (shown in Figure 7). This stitch is made by tying a half hitch into the bight and then knitting through it.

Combining stitches using annulus sums

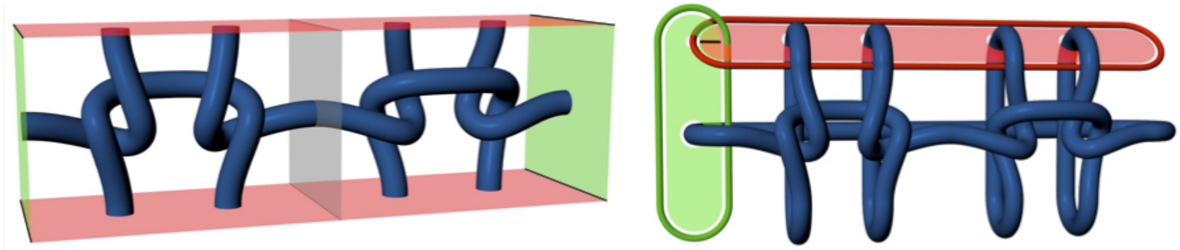


(a) *A disjoint link in* S^3. **(b)** *The two components of the link are joined by a band.* **(c)** *Band surgery swaps arcs along the edges of the band.*

Figure 8: *By joining 2-periodic knit stitches together in different ways we can generate the different fabrics in Figure 3.*

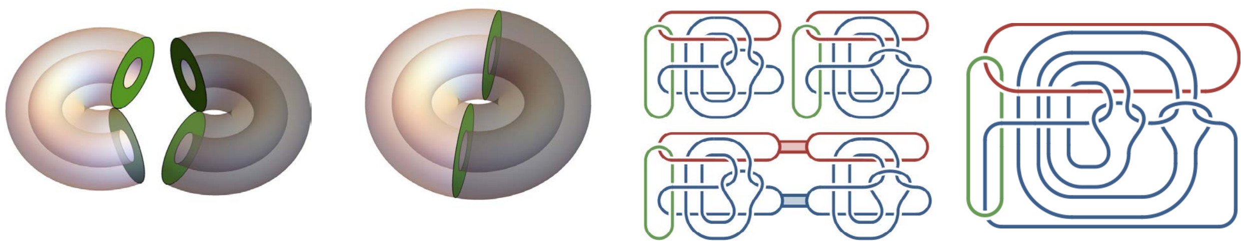
Now that we have constructed a standard position for textile knots in S^3, we need to construct an algebra for adding different stitch types together to create fabrics, as in Figure 3. In S^3, a *connected sum* of two disjoint knots $\mathcal{K}_1$ and $\mathcal{K}_2$, denoted by $\mathcal{K}_1\#\mathcal{K}_2$, joins $\mathcal{K}_1$ and $\mathcal{K}_2$ according to the following procedure: (1) take planar projections of two knots (Figure 8a), (2) find a rectangular patch where one pair of sides are arcs on each knot (Figure 8b) and (3) join the knots by deleting the two sides of the knot in the rectangle and connecting the other pair of sides (Figure 8c) [7]. Note the general procedure of changing the connectivity of a knot or link according the a rectangle (as in steps (2) and (3)) is called *band surgery*. This has many consequences for topological invariants. For instance, the Alexander polynomial V for $\mathcal{K}_1\#\mathcal{K}_2$, $V_{\mathcal{K}_1\#\mathcal{K}_2} = V_{\mathcal{K}_1}V_{\mathcal{K}_2}$ is a product of the Alexander polynomials for each individual knot, $V_{\mathcal{K}_1}$ and $V_{\mathcal{K}_2}$. This creates an algebra for building complexity of knots in S^3.

Each of the fabrics in Figure 3 are 2-periodic and can be made by combining knit and purl stitches either laterally — as in 1×1 ribbing shown in Figures 9, vertically — as in garter, or both — as in seed. Stockinette and reverse stockinette are represented by knots in $\mathbb{T}^2 \times I$ (or links in S^3). We would like to create

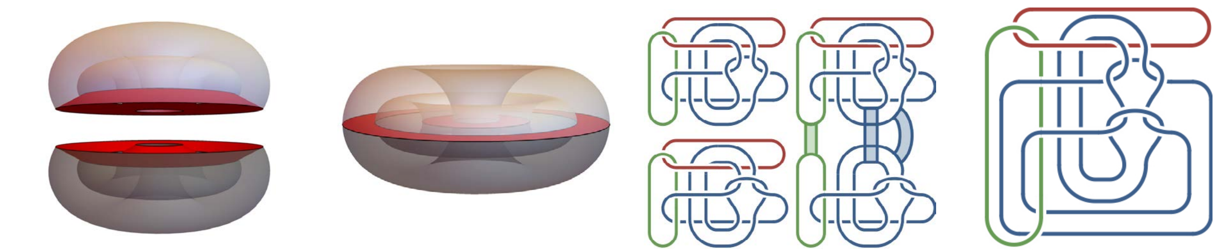


(a) *Two 2-periodic knit stitches are joined in the* $\mathbb{T}^2 \times I$ *model.* **(b)** *The same two stitches joined with the Hopf link in* S^3.

Figure 9: *Two stitches joined horizontally to create* 1×1 *ribbing.*



(a) *Adding stitches horizontally by cutting two tori along their meridians.*
(b) *The 3-manifolds are then glued together along the boundary annuli.*
(c) *In the S^3 picture, this can be algebraically realized using band surgery.*
(d) 1×1 *ribbing from a meridional annulus sum.*



(e) *Adding stitches vertically by cutting two tori along their longitudes.*
(f) *The 3-manifolds are then glued together along the boundary annuli.*
(g) *In the S^3 picture, this can be algebraically realized using band surgery.*
(h) *Garter stitch from an longitudinal annulus sum.*

Figure 10: *Annulus sum on the 3-manifold knot (or link) components defines the procedure for combining knit and purl stitches into more complicated 2-periodic textiles.*

a surgery on these knots (or links) that combines knit and purl stitches to create other 2-periodic textiles. We construct a method for combining stitches using an *annulus sum*. In an annulus sum, two 3-manifolds with toroidal boundary components (eg. the boundary of the tubular neighborhood of a knot) are each cut open along an annulus and then the annuli on the two original 3-manifold are identified. Figure 10a-d illustrates a longitudinal annulus sum, and Figure 10e-h demonstrate the meridional annulus sum. Consider two knit knots $\mathcal{K}_1$ and $\mathcal{K}_2$. These can either be viewed as two disjoint 3-manifolds $\mathbb{T}^2 \times I - \mathcal{K}_1$ and $\mathbb{T}^2 \times I - \mathcal{K}_2$ or as the 3-manifold created by the complement of two disjoint auxiliary links $\mathcal{L}_1$ and $\mathcal{L}_2$ in S^3. The annulus sum is a process to join the disjoint manifolds (or links in S^3) into a single knit knot, either both along their meridians or their longitudes. This process is not necessarily unique. For instance, if multiple link components pass through the annulus, the gluing operation can join them in any number of ways. In our operation, we can label components of the links in each of the original 3-manifolds and then insist they are identified during the gluing procedure. The result is a piecewise smooth 3-manifold which is the complement of the sum of two knitted stitches.

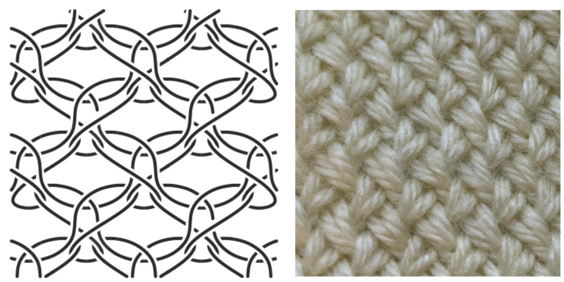
Adding stitches horizontally involves cutting two tori along their meridians in the $\mathbb{T}^2 \times I$ picture, or along the annulus bounded by the green component of the Hopf link in Figure 6g in the S^3 picture, see Figure 10a. In the $\mathbb{T}^2 \times I$, cutting each 3-manifold along along its meridian leaves two boundary annuli, punctured by the knit knot. In Figure 10b, each pair of annuli are glued together and the knit knot boundaries are identified. In the S^3 picture, the link complements are split along disks that span the meridional (green) component of their Hopf links. These disks are then glued together, identifying the punctures made by the knitted link components. This is equivalent to doing a pair of band surgeries on the links, shown in Figure 10c. The resulting knitted component of the link still has pairwise linking number one with the meridional (green) link component and is trivially linked with the longitudinal (red) component. Therefore, the knitted link component still has trivial homology around the meridian. Figure 10d shows a simple example of this is joining a knit link with a purl link along their meridians to create 1×1 ribbing.

Likewise, stitches can also be joined vertically. This process involves joining two thickened tori by cutting along their longitudes, as shown in Figure 10e. The resulting annular boundary components are joined together with the knitted (blue) link punctures identified (as in Figure 10f). In the S^3 picture, this involves cutting the 3-manifold along the disks spanning the longitudinal (red) component of the Hopf link and gluing the manifold together along those boundaries (see Figure 10g). This is equivalent to performing three band surgeries on the knit links. The vertical annulus sum adds a component to the link. This component corresponds to another knitted knot. Each of the two knitted components (blue) link with the meridional (green) component of the Hopf link, and they are trivially linked with each other and with the longitudinal (red) component of the Hopf link. Garter fabric can be created by joining a knit link with a purl link along their longitudes, as seen in Figure 10h.

Meridional and longitudinal annulus sums commute. The checkerboard lattice seen in seed fabric in Figure 3e can be created by first creating two tori longitudinally with garter links in them and joining them with a meridional annulus sum. The result is homeomorphic to the link generated by first creating two tori meridionally with 1×1 ribbed knots in them and then joining them together with a longitudinal annulus sum.

Some stitch patterns cannot be made using the annulus sum

There are other topologically allowed knitted stitches that respect the 2-periodic nature of textiles. These occur when the order of stitches within a given row is changed. In knitting, this is known as *cabling*. When stitches are moved, they can create either left leaning or right leaning crossings, when viewed with the wale direction vertically aligned. This creates an algebra of the rows that is analogous to the Artin braid group of n strands [9]. The generators of the braid group are denoted σ, where σ_i acts on strands i and $i+1$ to cross strand i over $i+1$; likewise, σ_i^{-1} crosses strand $i+1$ over i. For instance, the basketweave pattern in Figure 11a is generated on even rows by $\sigma_1\sigma_3\sigma_5...\sigma_n$ and on odd rows by $\sigma_2^{-1}\sigma_4^{-1}\sigma_6^{-1}...\sigma_{n-1}^{-1}$. The knotted topology of the knitted stitches also changes the algebraic structure of the braid group, such that for subsequent rows, it no longer has an inverse $\sigma_i\sigma_i^{-1} \neq 1$. This implies that the structure of the knitted equivalent of the braid group is not a group but a monoid. This is the set of transpositions of a string of n elements. Within a single row, any action of the braid group is valid until they are locked into place by the subsequent row of stitches.

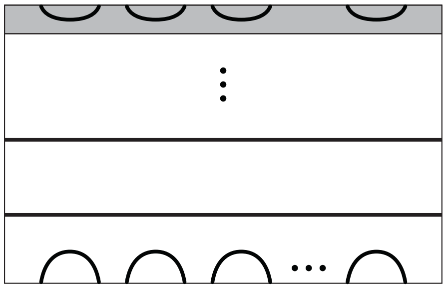


(a) *A (left) knot diagram for (right) basketweave fabric shows pairs of stitches that have been swapped, left leaning on odd rows and right leaning on even ones.*

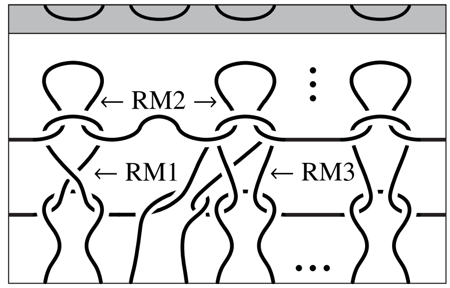
(b) *This idea can be extended to create braided cables often seen in aran sweaters.*

Figure 11: *Some knitable stitches cannot be made using the annulus sum.*

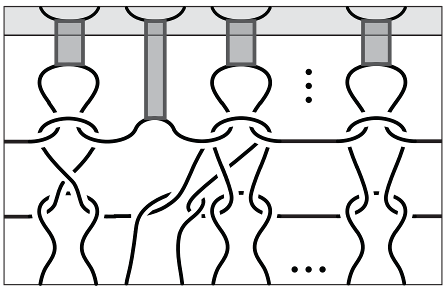
Cabling is a manipulation of stitches that can't be created by using the annulus sum process shown in Figure 10. We will construct a type of surgery on the manifold that allows us to create transpositions between elements. It is necessary to keep in mind that, as with braids, transpositions have a sense of orientation, either element i passes over $i+1$ or vice versa. We will incorporate these transformations into the connected sum algebra we have created for addition of different stitches into a period fabric. A single transposition, as in Figure 11a, involves interchanging two stitches. However, in more complicated cables, e.g. the braided cable in Figure 11b, two groups of consecutive stitches are interchanged, but this does not need to happen pairwise.



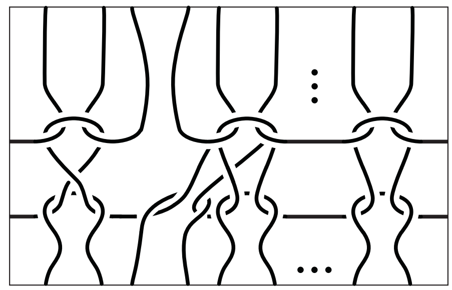
(a) *An $m \times n$ unknit begins with n disjoint, unlinked longitudinal circles and m disjoint circles with trivial homology.*



(b) *Ambient isotopy between bights of the disjoint circles and the longitudinal circles.*



(c) *Band surgery joins the m disjoint circles to one of the n longitudinal circles.*

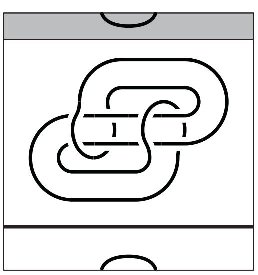


(d) *An $m \times n$* swatch *in $\mathbb{T}^2 \times I$.*

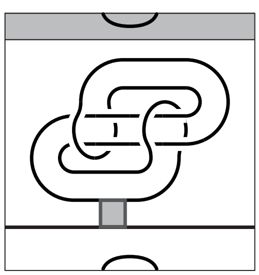
Figure 12: *Construction of and $m \times n$* swatch.

The swatch construction of all knitted textiles

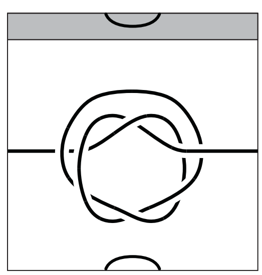
Although these more complicated multi-stitch objects cannot be constructed from basic knit and purl elements using annulus sums, they do fit into our framework of links in $\mathbb{T}^2 \times I$. This construction, which we call a *swatch*, is an entirely general construction that can be used to create any knitted motif. A swatch begins with an n-stranded *unknit*, made from n disjoint circles along the longitude of the torus and m disjoint circles with trivial homology, see Figure 12a. Figure 12b shows that bights of each of the m circles interacting via ambient isotopy with one or more of the n longitudinal strands. These strands are now able to interact with one another via ambient isotopy. Since this is a planar projection, we can use Reidemeister moves to create these ambient isotopies. In Figure 12b, we label the specific Reidemeister moves necessary to create the example motif. Note that this procedure does not change the pairwise linking number of any of the circles. Finally, each of the m circles are joined by band surgery to bights in the last longitudinal strand (Figure 12c) to create the final swatch in $\mathbb{T}^2 \times I$ (Figure 12d). As the swatches live in $\mathbb{T}^2 \times I$, an $k \times n$ swatch and an $l \times n$ swatch can be joined via a meridional annulus sum to create a $(k + l) \times n$ swatch. Likewise, $m \times k$ and $m \times l$ swatches can be joined longitudinally to cre-



(a) *This $1x1$ swatch diagram contains an additional disjoint knot.*



(b) *It is connected to the longitudinal component of the swatch.*



(c) *The resulting ribbon knot is a motif that cannot be created by knitting.*

Figure 13: *Shows a counterexample to the claim that all ribbon knots can be knitted.*

ate an $m \times (k + l)$ swatch.

All of the objects we have considered thus far fit into this swatch construction. The basic knit and purl are types of 1×1 swatch, as is the cow-hitch. 1×1 ribbing is a 2×1 swatch, while garter is a 1×2 swatch. The basketweave structure in Figure 11a is a 4×2 swatch. This construction shows that all knitted link components are ribbon. However, we can easily show that not all ribbon knots are knitable. For example, we can take the connected sum of a ribbon knot with any of the n longitudinal circles, eg. Figure 13. The resulting knot, when viewed as a component of knitted link joined with the Hopf link in S^3 is ribbon, but it is no longer knitable.

Summary and Conclusions

Here, we presented a topological framework for 2-periodic knitable structures as knots in $\mathbb{T}^2 \times I$ (or as a link in S^3). Using meridional and longitudinal annulus sums, we can join different primitive knit elements together to create more complex textiles, including 1×1 ribbing, garter and seed fabrics. Knits allow for multiple stitches between rows to interact with each other in non-pairwise ways, thus annulus sums cannot create all possible knits. We define the swatch as a way to construct all knitable objects in $\mathbb{T}^2 \times I$. Multiple swatches can be joined together using the annulus sum to create more textiles.

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