Non-Euclidean Billiards in VR

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Abstract

Virtual reality simulations connect not only with our conscious minds, they also completely hijack our subconscious understanding of our environment, giving us a convincing illusion of playing billiards in 3-dimensional hyperbolic space, in Euclidean space, and on a hypersphere. Even experienced geometers may find some surprises there.

Experiencing Curved Space

Experiencing 2-dimensional surfaces—the sphere, the Euclidean plane, and the hyperbolic plane—is relatively easy: we can make fleece models of them and hold them in our hands (Figure 1). With the arrival of Virtual Reality (VR) systems, experiencing curved 3-dimensional space is now equally easy: we can put on a VR headset and *be* in the curved space.

Of course just being in a curved space isn't enough—once we're there we need something to *see* and most importantly something to *do*. The present project implements a billiards game in a hypersphere (3-sphere), in Euclidean 3-space, and in hyperbolic 3-space (Figure 2). I chose billiards for several reasons:

- There's essentially no learning curve. Players can put on the VR headset and immediately start playing.
- The game gets the player thinking about the billiard table's geodesics (straight lines) almost immediately, with no conscious effort required.
- The player can walk around freely in the curved space (for example to take a shot from the other side of the table), yet feels no temptation to wander away from table, which would risk walking into walls or furniture back in "real reality".

Even seasoned geometers may encounter surprising optical effects upon their first virtual-reality visit to curved space (for sure I did). The present article explains the geometry behind a few of these effects. A separate, complementary article [1] explains the mathematics underlying the curved space VR software itself.

(a) *Sphere* **(b)** *Euclidean plane* **(c)** *Hyperbolic plane*

Figure 1: *Fleece surfaces*

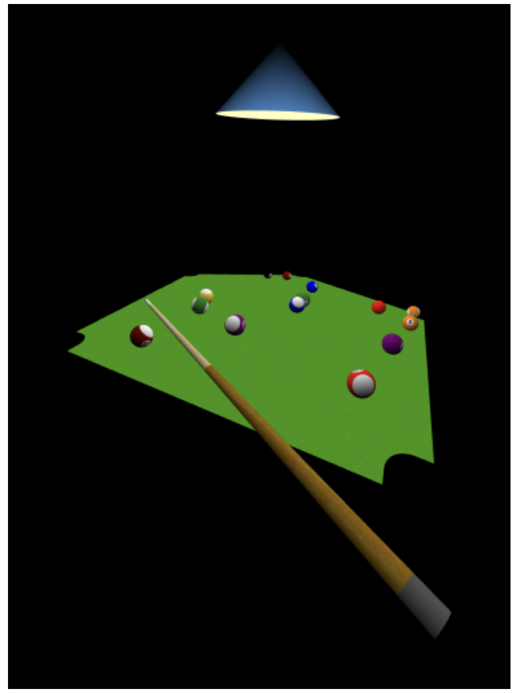


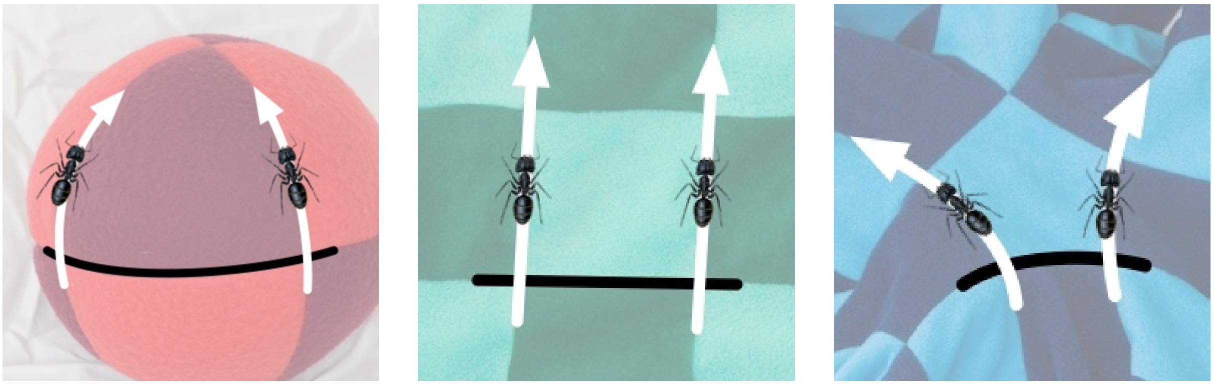
Figure 2: *Billiards in Hyperbolic 3-Space*

Seeing Curvature

How can a person see the curvature of the space that she finds herself in? The easiest way is to look at how geodesics behave. A *geodesic* is a straight line: on a 2-dimensional surface it's a path that bends neither leftward nor rightward; in a 3-dimensional space it's a path that bends neither leftward, rightward, downward nor upward.

For simplicity, let's start with the 2-dimensional surfaces. Draw a black geodesic (a straight line) in the Euclidean plane, and then draw two white geodesics perpendicular to it (Figure 3(b)). If two ants start walking along the two white geodesics, the distance between them will stay constant as they walk. Now repeat the experiment on a sphere: draw a black geodesic (a great circle) and two white geodesics perpendicular to it (Figure 3(a)). This time, as the ants walk along white geodesics, the distance between them decreases, until they finally meet at the north pole. This phenomenon is called *geodesic convergence* and is characteristic of spherical geometry. By contrast, if you repeat the experiment on a hyperbolic plane (Figure 3(a)), the distance between the ants increases as they walk along. This is called *geodesic divergence* and is characteristic of hyperbolic geometry.

The same principle works to see the curvature of a 3-dimensional space. For an example, you may download the *Curved Spaces* app [2] and from its Space menu choose Mirrored Dodecahedron. If you fly forwards in that space (Figure 4), the behavior of its geodesics will show you whether it's spherical, Euclidean, or hyperbolic.



(a) *Geodesic convergence* **(b)** *Parallel geodesics* **(c)** *Geodesic divergence*

Figure 3: *Geodesics reveal curvature.*

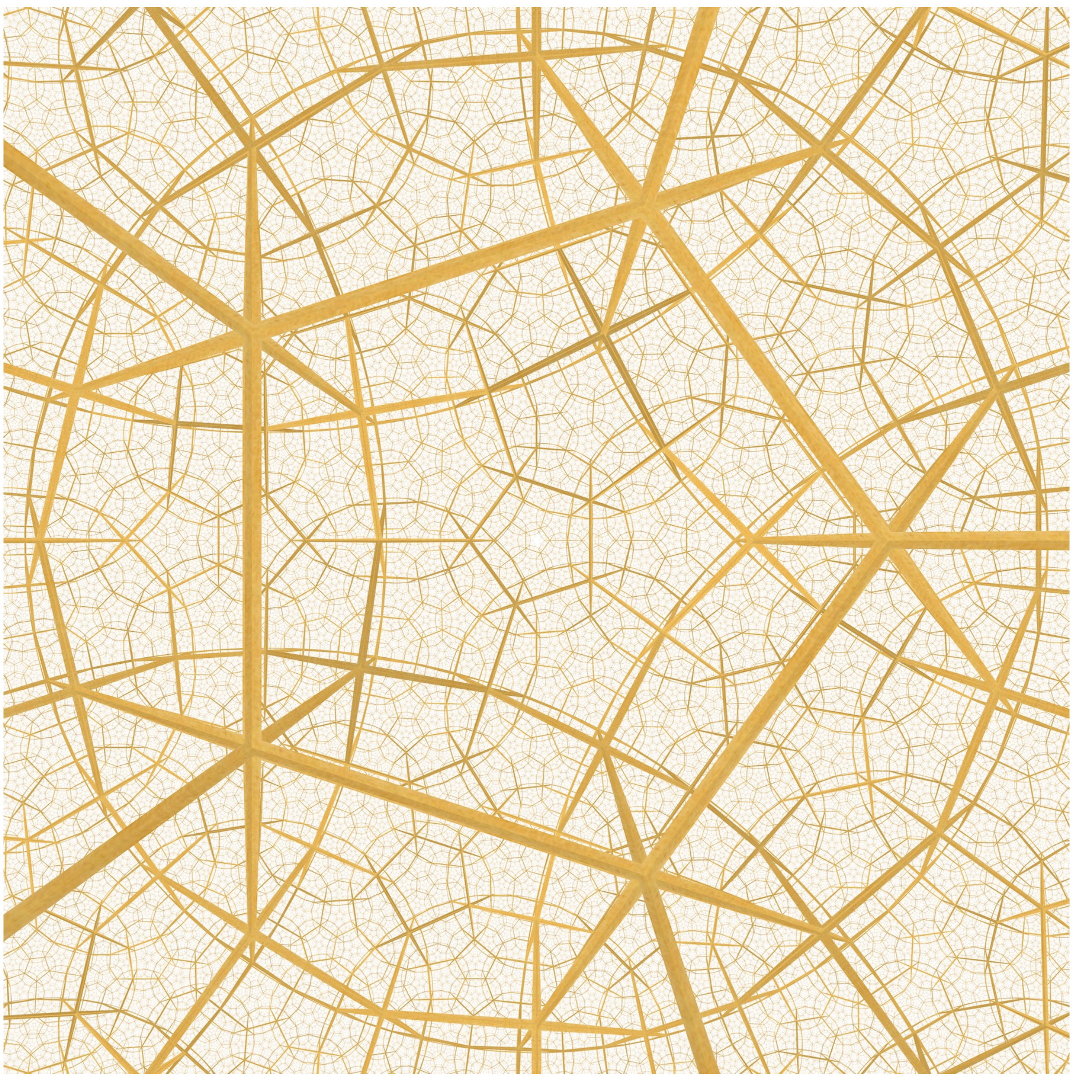


Figure 4: *Looking at the geodesics can tell us whether this space is spherical, Euclidean, or hyperbolic.*

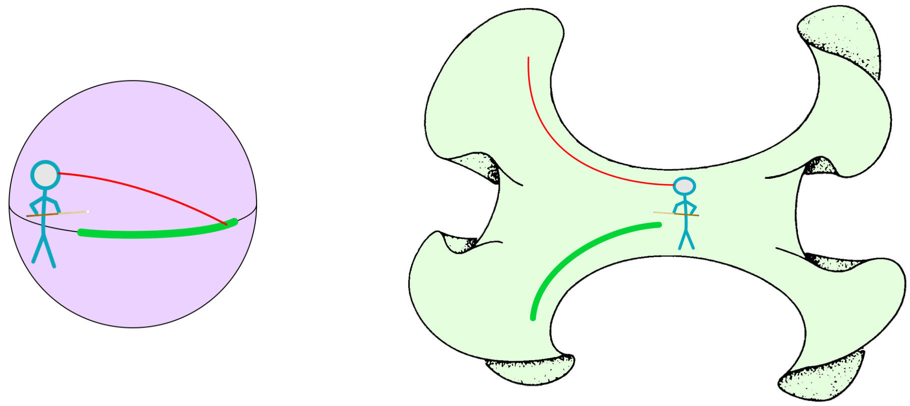


Figure 5: *Geodesic convergence (left) or divergence (right) gives a Euclidean-born tourist the illusion that table tilts upward or downward.*

Billiard Table Seems Tilted

When playing billiards in a curved space, the first effect that many players notice is that the billiard table seems to slope upwards (in spherical geometry) or downwards (in hyperbolic geometry). To understand why, first consider the spherical case (Figure 5(left)). When the player looks straight ahead, her line of sight and the table surface both extend perpendicularly outward from her body. Because of geodesic convergence, her line of sight hits the table at a point roughly one meter in front of her. A billiards player who was born and raised in spherical space would correctly perceive the table to be level. But a “tourist”, visiting from Euclidean space, would see the table’s far edge sitting “at eye level” and, based on her years of experience in Euclidean space, would falsely conclude that the table tilts upwards.

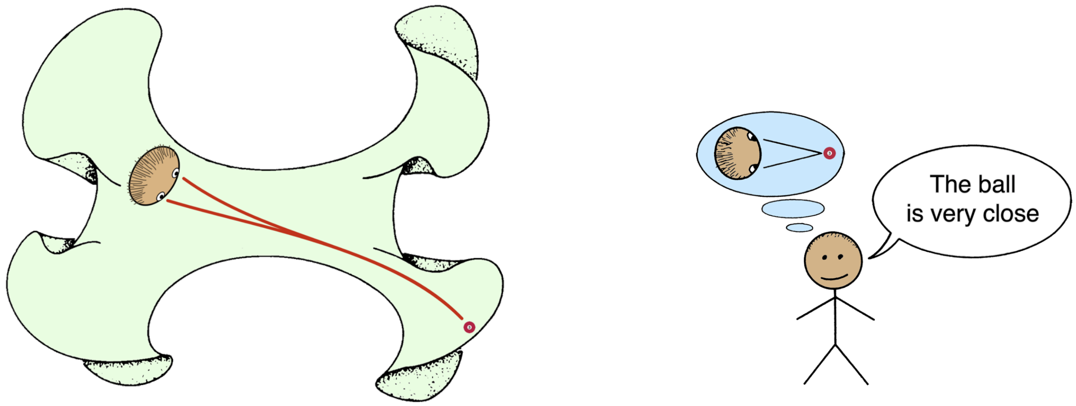
In hyperbolic space, by contrast, when a player looks straight ahead, her line of sight goes nowhere near the table (Figure 5(right)). A billiards player who was born and raised in hyperbolic space would correctly perceive the table to be level. But a Euclidean-born tourist would falsely conclude that the table slopes downward.

In both cases, the root of the problem is that the tourist’s mental model of space is Euclidean, and she forces her spherical or hyperbolic perceptions onto that Euclidean mental model.

More precisely, the root of the Euclidean-born tourist’s misperception lies in the concept of a *horizon*. In Euclidean geometry, our horizon is a great circle on our visual sphere. Moreover, all horizontal planes share that same horizon circle. But those facts are true *only* in Euclidean geometry. In hyperbolic geometry, a plane’s “horizon” is typically a lesser circle (not a great circle), and a horizontal plane through your ankles has a smaller horizon circle than a horizontal plane through your waist! In spherical geometry, there’s no concept of horizon at all—every “plane” (that is, every great sphere) fills your whole visual sphere. The only exceptions are planes that pass through your eye.

Stereoscopic Vision

Because of geodesic convergence, a player has to look slightly cross-eyed to focus on a distant billiard ball in hyperbolic space. A born-and-raised native of hyperbolic space would be accustomed to this cross-eyedness and would correctly perceive the ball to be far away (Figure 6(a)). By constrast, a tourist from Euclidean space, who’s accustomed to looking straight ahead to focus on distant objects (with no cross-eyedness), would mistakenly perceive the ball to be close by (Figure 6(b)). Indeed the Euclidean-born tourist mistakenly perceives the whole infinite hyperbolic space as sitting inside a finite ball of radius roughly 1 meter. As the



(a) *A native inhabitant of hyperbolic 3-space correctly perceives the ball to be far away*

(b) *A Euclidean-born tourist's erroneous impression of hyperbolic 3-space*

Figure 6: *Native view vs. tourist view in hyperbolic space*

tourist walks around, the entire contents of the whole universe seem to move along with him, all trapped inside that finite ball.

The Euclidean-born tourist's impression of spherical space is even more distressing. As a consequence of the geodesic convergence, when the tourist looks at ball #1 (yellow) in Figure 7, he sees it with less cross-eyedness (that is, with a smaller *vergence angle*) than he would if he were in Euclidean space; as a result, he misperceives the yellow ball as sitting further away from him than it really is. Ball #2 (blue) sits 90° away from the tourist in Figure 7, so he must look straight ahead to see it (vergence angle = 0) and therefore misperceives it as sitting infinitely far away. Ball #3 (red, but on back side of sphere) is the worst of all: to focus on it, the tourist must look "walleyed" (negative vergence angle), with each eye looking off to the side! While testing the Non-Euclidean Billiards software, I found that the human visual system really can look slightly walleyed and still focus on an object. But, as you might expect, the experience is uncomfortable and vaguely distressing. My guess is that using the VR system this way for an extended period of time would give the player a headache, or worse.

To spare the player from such misleading—and potentially painful—perceptions of hyperbolic and spherical space, the Non-Euclidean Billiards app automatically adjusts the vergence angle so that the player sees each point in the space at the same distance that a native inhabitant of that space would see it. In other words, when you play Non-Euclidean Billiards you will perceive all the balls, the pockets, the cue stick, etc. to be sitting at their true distances from you.

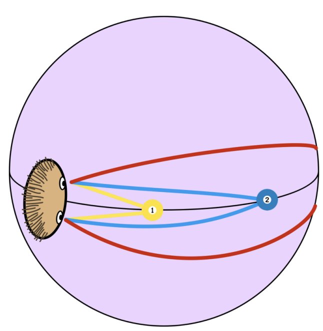


Figure 7: *The Euclidean-born tourist perceives ball #2 to be infinitely far away, because he sees it best with both his eyes looking straight ahead.*

Why Does Stuff Still Seem Concave/Convex?

Even with the adjustment described immediately above, which lets you see all objects at their true distances, you might be surprised to find that your Euclidean-born-tourist brain nevertheless perceives the cue stick and the table to be concave in spherical space, and convex in hyperbolic space. The reason is that human vision depends on three factors

- the vergence angle with which your eyes see the object,
- how your brain converts that vergence angle to a perceived distance,
- how some other part of your brain integrates the perceived distances to construct a mental model of the 3-dimensional space

and it's the third factor where your Euclidean-born-tourist brain makes its mistake.

To understand what goes wrong, imagine a cue stick sitting 1 meter in front of your face in Euclidean space (Figure 8). Point C (the stick's midpoint) is closest to you, points B and D are slightly further away, and points A and E are further still. If you repeat this experiment in spherical space, you'll observe almost the same thing…except that distances increase a little more slowly as you shift your attention from point C to D and thence to E. And, by contrast, the distances increase a little more rapidly in hyperbolic space.

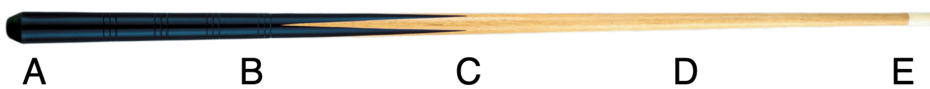
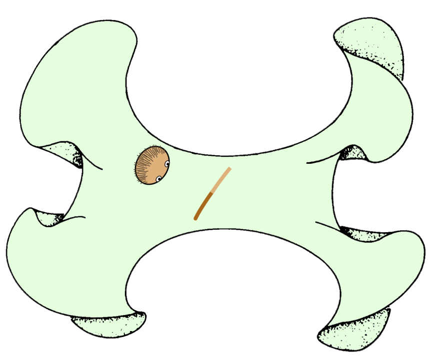


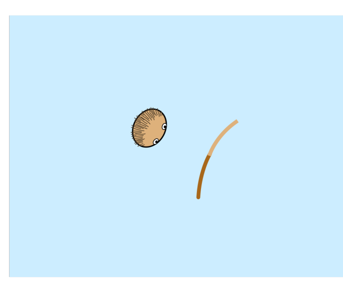
Figure 8: *The cue stick's midpoint C is closest to you. How fast the distance increases as you shift your attention from point C to point D and thence to E depends on the curvature of the ambient space.*

So when you put on the VR headset and play hyperbolic billiards, even though the app is showing you the correct hyperbolic distance to every point on that cue stick, your brain chooses to misinterpret that data as a curved stick in a flat space (Figure 9(b)), rather than as a straight stick in curved space (Figure 9(a)). If, while you're playing billiards, you shut one eye, the effect disappears immediately: the cue stick instantly looks straight! If you spent enough hours playing hyperbolic pool (with both eyes open!), your brain might start to see the cue stick as straight, but I think that would be an unhealthy experiment. Please don't try it.

Exercise for the reader: If a person born and raised in hyperbolic space were to come visit us here in Euclidean space, how would she perceive one of our cue sticks?



(a) *A native inhabitant of hyperbolic 3-space perceives a straight cue stick in a curved space.*

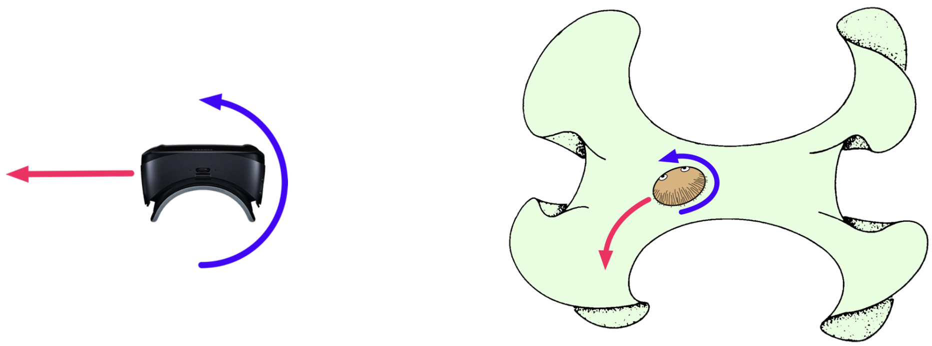


(b) *A Euclidean-born tourist incorrectly perceives a curved cue stick in a flat space.*

Figure 9: *The native and the tourist see the same cue stick in the same hyperbolic space, yet they interpret what they see very differently.*

What Would Curved Space Feel Like?

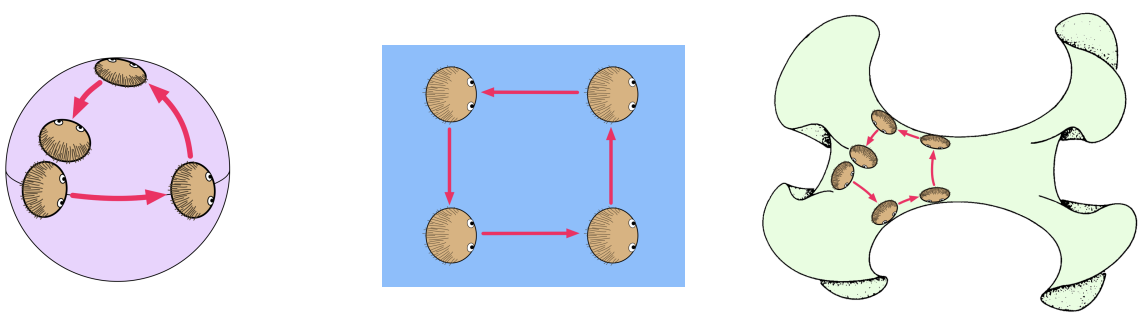
If you could visit a curved space—not just a VR simulation but the real thing—it would not only look different from Euclidean space, it would also *feel* different. Surprisingly, our story here starts with headset tracking (Figure 10). At the most basic level, when the headset moves a centimeter to its left in physical Euclidean space, the observer moves a centimeter to her left in the virtual hyperbolic space. When the headset rotates 2°, the observer rotates 2°, and so on. But dynamically things are more complicated, due to *holonomy* effects.



(a) *Flat physical space* **(b)** *Curved virtual space*

Figure 10: *Headset tracking*

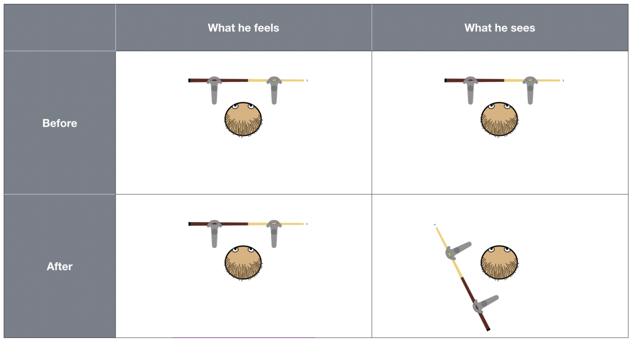
What's holonomy? An observer who slides around a loop in Euclidean space (being careful not to spin on her own axis as she slides around!) comes back unrotated (Figure 11(b)). But an observer who slides around a loop on a sphere (again taking care not to spin on her own axis) comes back rotated relative to how she started (Figure 11(a)). This effect is called *positive holonomy*. An observer who slides around a loop in hyperbolic space also comes back rotated (Figure 11(c)), but in the opposite sense (clockwise or counterclockwise) to the sense in which she slid around the loop. This is called *negative holonomy*.



(a) *Positive holonomy* **(b)** *Zero holonomy* **(c)** *Negative holonomy*

Figure 11: *Holonomy*

If the Billiards app tried to map motions of the physical headset directly into the curved space (Figure 10), the player would encounter a problem with the coherence of his own body. Say, for example, he's standing in the physical play area, holding two physical VR controllers, which in turn hold the virtual cue stick. He *feels* the controllers in front of him, and he *sees* the cue stick in front of him, and all is well (Figure 12(a), top row). But if, while keeping his shoulders perfectly still, he starts sliding his head around in a small circle (as in Figure 11(a) or Figure 11(c), but around a much smaller loop, for comfort's sake), then because of the holonomy his head would come back rotated relative to the rest of his body in the curved



(a) *Uncorrected VR simulation*

(b) *Required neck torque*

Figure 12: *Holonomy can make a player's head spin.*

space, even though in the physical Euclidean play area it remains unrotated. So now he'd *see* the cue stick sitting off to the side somewhere, even though he'd still *feel* it sitting right in front of him (Figure 12(a), bottom row).

If you visited hyperbolic space, would that really happen? No, of course not. What you'd see and what you'd feel would stay perfectly consistent. But as you moved your head around in a small circle, you'd feel a slight torque in your neck. You'd have to use your neck muscles to counter that torque to keep your head looking forwards (Figure 12(b)). It's like what happens when you hold a spinning gyroscope in your hand and try to rotate it. You can't just pick an axis and freely rotate about it, because the gyroscope will want to rotate about a perpendicular axis. To force the gyroscope to rotate about your chosen axis, your hand needs to apply an extra torque to counter that "mysterious" gyroscopic effect. Similarly, as you move your head around in hyperbolic space, your neck needs to apply an extra torque to counter the "mysterious" holonomy effect.

The VR headset has no force feedback system (thank goodness!), so the Billiards app instead simply pretends that your neck is providing the required torque to keep your head and your body aligned.

Summary and Conclusions

Static images cannot convey what curved spaces are like, nor can ordinary computer animations. Only a virtual reality simulation gives you the experience of *being* in a hyperbolic or spherical world. I look forward to welcoming Bridges participants to come play billiards and experience these wonderful spaces for themselves. Most of all, I hope some artists will be inspired to create sculptures that are truly non-Euclidean. I will happily provide source code and technical assistance to anyone who wishes to explore the possibilities.

Acknowledgements

I thank Sabetta Matsumoto for her generous and patient help getting me started with virtual reality.

References

[1] J. Weeks. "Virtual Reality Simulations of Curved Spaces." *Experimental Mathematics*, in preparation.

[2] J. Weeks. *Curved Spaces*. http://www.geometrygames.org/CurvedSpaces.