Hyperbolization on a Squircular Continuum

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Abstract

A squircle is an intermediate shape between the square and the circle. In this paper, we propose equations for a new type of squircle and present an algorithm to map the Poincaré disk to this shape. Furthermore, we use this algorithm as a framework for hyperbolic animation, whereupon we morph the Poincaré disk into a square.

Introduction

M.C. Escher was interested in enclosing infinite patterns within the confines of a finite region. He succeeded in doing this after H.S.M. Coxeter pointed him to a mathematical construct known as the Poincaré disk. This resulted in four of his famous "*Circle Limit"* woodcuts [2]. Our goal in this paper is to follow on Escher's footsteps and produce hyperbolic artworks that are spiritual successors of his works.

Loosely speaking, a squircle is a hybrid of the square and the circle. There are actually several different types of squircles which we will discuss in a mathematically precise manner later. In this paper, we will introduce equations for a new type of squircle. Moreover, we will present an algorithm to enclose infinite patterns within the confines of this shape. As a preview, we show some of our results in Figure 1.

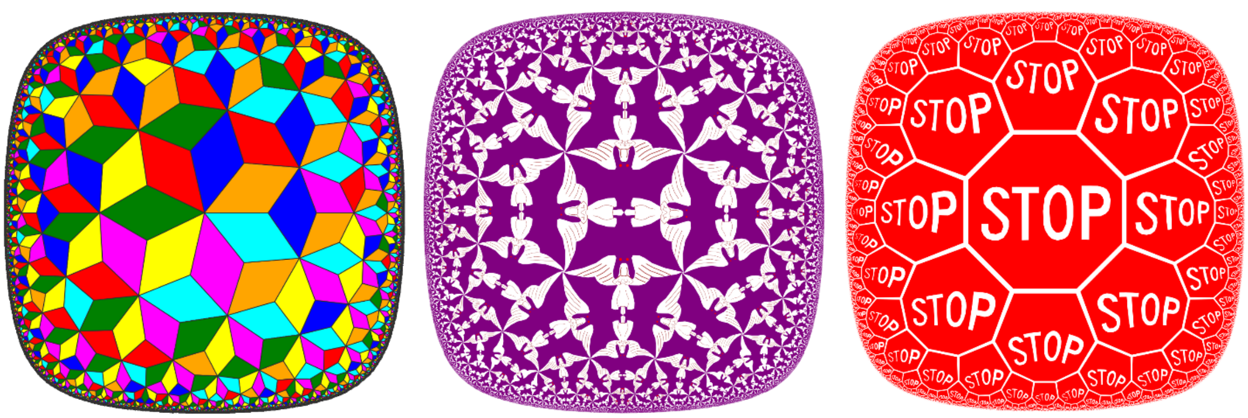

Figure 1: *Hyperbolic patterns enclosed inside a squircle*

Escher himself was interested in squircles. This is evidenced by his *"Fishes and Scales"* woodcut [3] shown in Figure 2. Although we are not experts in art history, we believe that Escher was a pioneering artist in his explicit use of the squircle in this artwork. Furthermore, we have been unable to find any other historical or famous paintings enclosed inside a squircle.

"Fishes and Scales" illustrates a multitude of fishes enclosed inside a squircle. The woodcut pattern is not quite hyperbolic, but it has all the tell-tale idiosyncrasies of Escher's handiwork. Specifically, it shows many interlocked fishes going in opposite directions without overlap. This is a recurring theme in many of Escher's works and served as the inspiration for this paper. Upon close examination of Escher's interlocking patterns, we wondered if there is a way to convert his "*Fishes and Scales*" into artwork that is reminiscent of his "*Circle Limit"* woodcuts.

As a matter of fact, we did figure out a way to hyperbolize Escher's "*Fishes and Scales*". This is shown in Figure 2. As an homage to Escher, we shall refer to this artwork as *"Squircle Limit"*. The rest of this paper will discuss how we went about creating this piece.

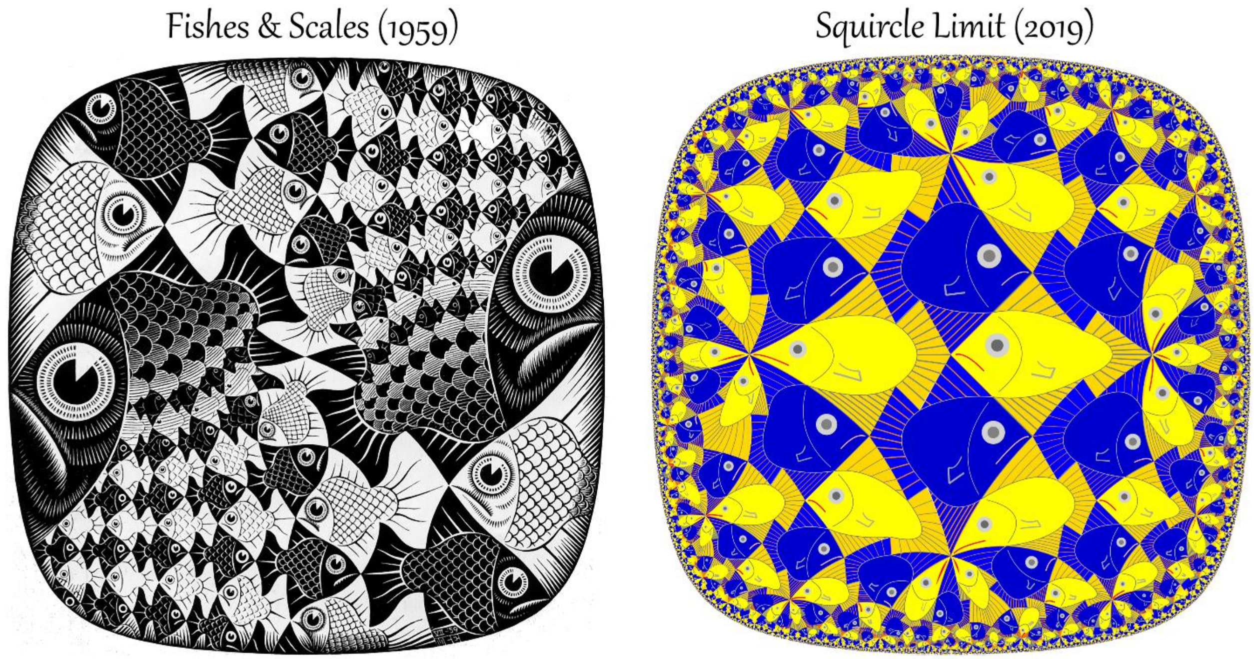

Figure 2: *A hypermodern homage to Escher*

A Menagerie of Squircles

As previously mentioned, a squircle is a geometric hybrid of the circle and the square. Although this definition is not mathematically precise, there are ways to formalize this with the use of equations. But first, we would like to mention that there are several different types of squircles [8]. Of particular interest to us are squircles that can interpolate between the circle and the square. In this section, we shall briefly go over three of these. In the next section, we shall introduce another one.

Square with Rounded Corners. This is by far the most common type of squircle that people encounter in their daily lives. A wide variety of regular items such as dinner plates and road signs use this as an underlying shape. The popularity of electronic products from Apple Inc. helped foster the sleek image of this shape. Mathematically speaking, this shape is fairly simple. It is just a concatenation of four straight line segments from a square with four quarter circular arcs at the corners. This shape is illustrated in Figure 3.

Lamé Squircle. Most students of analytic geometry in the Cartesian coordinate system are already familiar with the superellipse. The Lamé squircle is just a special case of the superellipse with no eccentricity. It was originally studied by the Gabriel Lamé in 1818. The Lamé squircle has the algebraic equation: $|x|^P + |y|^P = r^P$.

There are two parameters in this equation: p and r. The power parameter p is an interpolating variable that allows one to blend the circle with the square. When $p=2$, the equation produces a circle with radius r. As $p \rightarrow +\infty$, the equation produces a square with a side length of $2r$. In between, the equation produces a smooth planar curve that resembles both the circle and the square.

Fernandez-Guasti Squircle. In 1992, Manuel Fernandez-Guasti discovered a plane algebraic curve [4,8] that is an intermediate shape between the circle and the square. His curve is represented by the quartic polynomial equation: $x^2 + y^2 - \frac{s^2}{r^2} x^2y^2 = r^2$. The equation includes an interpolating parameter s that specifies the squareness of the shape. When $s = 0$, the equation produces a circle centered at the origin. When $s = 1$, the equation produces a square, also centered at the origin.

The Fernandez-Guasti squircle is algebraically simpler than the Lamé squircle because of its low degree polynomial equation. In contrast, the Lamé squircle has unbounded polynomial exponents that make it algebraically unwieldy. Figure 3 shows all the squircles mentioned in this paper. Note that it is not easy to tell these shapes apart from each other visually at various squareness values.

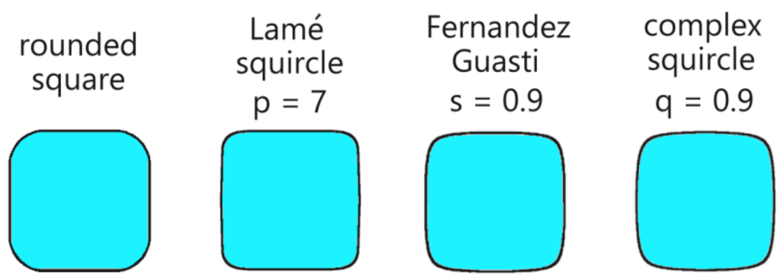

Figure 3: *Four types of squircles mentioned in this paper*

A Novel Type of Squircle

In this section, we will introduce yet another type of squircle. The main motivation behind this squircle is that it has some special properties with regards to the Poincaré disk. This will be discussed in detail later. We shall refer to this novel shape as the *complex squircle*. The reason behind this naming comes from the fact that the defining equations of this shape use complex variables. Furthermore, the equations require a special function F known as the incomplete Legendre Elliptic Integral of the 1st kind.

The complex squircle shall be defined in terms of parametric equations. Before we do this, let us first define an auxiliary complex-valued function Ω with input variable $\boldsymbol{w}$

$$\Omega(\boldsymbol{w}) = 1 - i - \frac{\sqrt{-2i}}{K_e} F\left(\cos^{-1}\left(\boldsymbol{w}\sqrt{i}\right), \frac{1}{\sqrt{2}}\right)$$

Using this complex-valued auxiliary function, the parametric equations for the complex squircle are

$$x(t) = Re\left[\,\Omega\left(qe^{it}\right)\right] \frac{r}{\Omega(q)} \qquad\qquad y(t) = Im\left[\,\Omega\left(qe^{it}\right)\right] \frac{r}{\Omega(q)}$$

At this point, there are a handful of variables and parameters that need to be explained. Let us start with the simplest ones. The variable q is a squareness parameter analogous to the ones provided with the Lamé and Fernandez-Guasti squircles. When $q = 0$, the complex squircle becomes a circle with radius r. When $q = 1$, the complex squircle becomes a square with side length $2r$. In between, the shape is a smooth curve that interpolates between the circle and the square. This is illustrated in Figure 4.

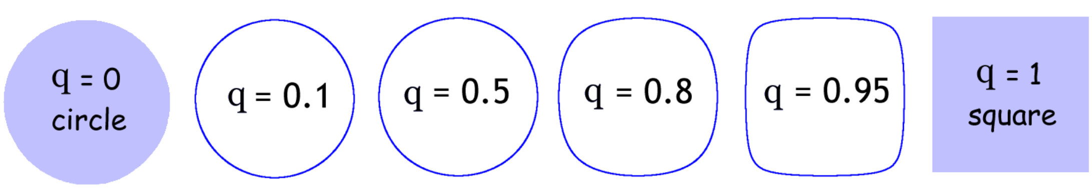

Figure 4: *The complex squircle at varying squareness values.*

The parameter t is the standard variable of the parametric equation. It can take on any value between 0 and 2π to span the full squircular curve. The numerical constant K_e is approximately 1.854 in value. We will give more details on it shortly. The other numerical constants in the equations are e and i, which need no introduction. In essence, they highlight the fact that the equations are complex-valued.

The complexities of the equations do not end there. The $\sqrt{\pm i}$ factors that show up in the auxiliary function are just a form of mathematical shorthand in order to keep the equations compact. Mathematically, $\sqrt{\pm i}$ represents a rotation of 45º in the complex plane, that is,

$$\sqrt{\pm i} \;=\; \tfrac{1}{\sqrt{2}}(1 \pm i) \;=\; \cos\frac{\pi}{4} \pm i\sin\frac{\pi}{4} \;=\; e^{\pm i\frac{\pi}{4}}$$

Legendre Elliptic Integral. The complex auxiliary function Ω has the special function F at its core. Mathematically speaking, the incomplete Legendre elliptic integral of the 1st kind is a two-parameter function defined as

$$F(\phi, k) = \int_0^{\phi} \frac{1}{\sqrt{1 - k^2 \sin^2 t}}\, dt$$

Note that this integral cannot be simplified using any of the standard techniques covered in freshman calculus classes. This integral was originally studied in the context of measuring the arc length of an ellipse. That is the reason why it is called an elliptic integral. The two arguments that come with this function are also intimately tied to the ellipse. The first parameter ϕ is an angular parameter. It originates from the angle subtended by an arc of the ellipse. The second parameter k is closely related to the eccentricity of the ellipse. Using this definition of the incomplete Legendre elliptic integral of the 1st kind, we can give a precise value for the K_e constant. Simply put, $K_e = F(\frac{\pi}{2}, \frac{1}{\sqrt{2}})$.

The complex-valued version of the special function F is required in the equations of the complex squircle. This complex variant is well-defined mathematically and can have complex-valued parameters. The full formula for this is provided by L.M. Milne-Thompson in the classic AMS55 reference book [1].

The Conformal Square. In the Bridges 2016 conference, Fong [5] introduced a conformal method for mapping patterns inside the Poincaré disk to a square, thereby establishing a square model of the hyperbolic plane. This model is now more commonly known as the *conformal square*. The method is based on the theory of Schwarz-Christoffel transformations in the complex plane.

Figure 5 shows an example hyperbolic pattern of angels & devils converted into the conformal square. The equation for this mapping is included with the figure. The complex variable $\boldsymbol{w}$ represents points inside the Poincaré disk. The complex variable $\boldsymbol{z}$ represents points inside the conformal square.

It is not a coincidence that the conformal square mapping equation looks identical in form to the auxiliary function Ω previously defined with the complex squircle. In fact, the conformal square mapping equation can be rewritten as $\boldsymbol{z} = \Omega(\boldsymbol{w})$. To put it simply, the auxiliary function Ω maps points inside the Poincaré disk to points inside the conformal square.

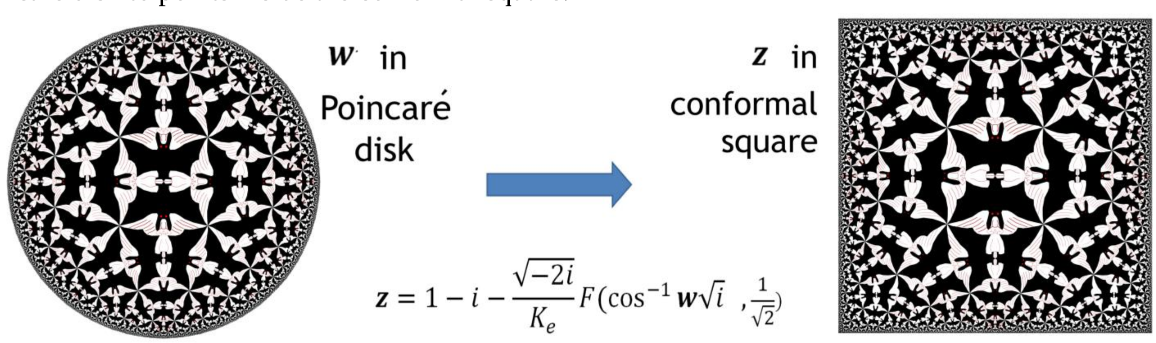

Figure 5: *Mapping the Poincaré disk to the conformal square*

The Squircular Continuum. The circular disc can be considered as a continuum of concentric circles with a radius growing from 0 to 1. This circular continuum is a key ingredient to how the complex squircle was derived. Figure 6 shows a circular disc subdivided into concentric annular rings of different colors. This circular disc is then mapped to the conformal square in the same figure. The resulting square is still subdivided by concentric rings, but the enclosing contours are not quite circular. In fact, the enclosing contours get increasingly square-like as they approach the rim of the square. Consequently, it is logical to ask what sort of shape encompasses the boundaries of the concentric rings inside the square.

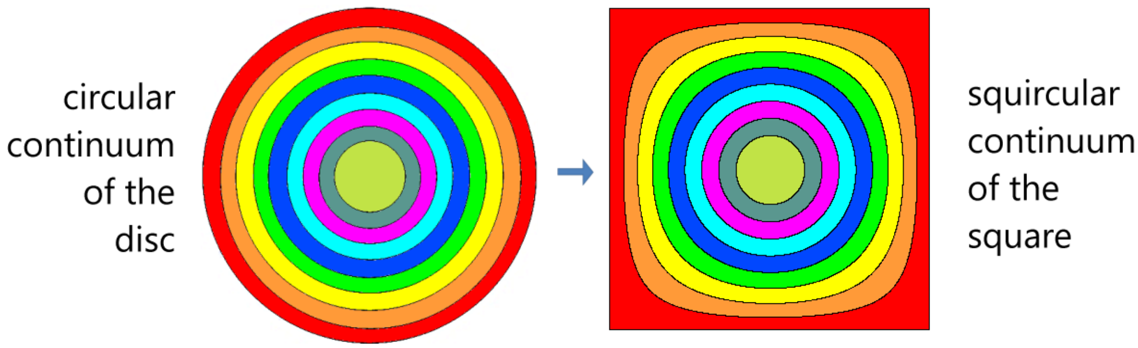

Figure 6: *Visual representation of the squircular continuum*

One can intuitively surmise from Figure 6 that the concentric shapes inside the square are some sort of squircular curve. Indeed, these concentric shapes exhibit all the defining characteristics of a hybrid shape between the circle and the square. This is how we derived the complex squircle. The continuum of concentric circles inside the circular disc corresponds to a continuum of concentric squircles inside the square. After normalizing these concentric squircles in size, we get the complex squircle. To summarize, the complex squircle is the shape resulting from mapping of a circular contour from inside a disc to the conformal square.

Mapping the Poincaré Disk to a Squircle

Having defined and derived the complex squircle in the previous section, we can now discuss how to map the Poincaré disk to this shape. The key idea lies in the squircular continuum shown in Figure 6. Observe that inner circular contours get mapped to squircles inside the square. Hence, by scaling down our input circular pattern, we are able to map it into a complex squircle. This can be done by simply using the equation in Figure 5. Of course, the resulting squircle will be similarly shrunken, so it has to be enlarged to fill the frame of the square. This algorithmic pipeline is shown in Figure 7 using a smiley emoji as the input pattern.

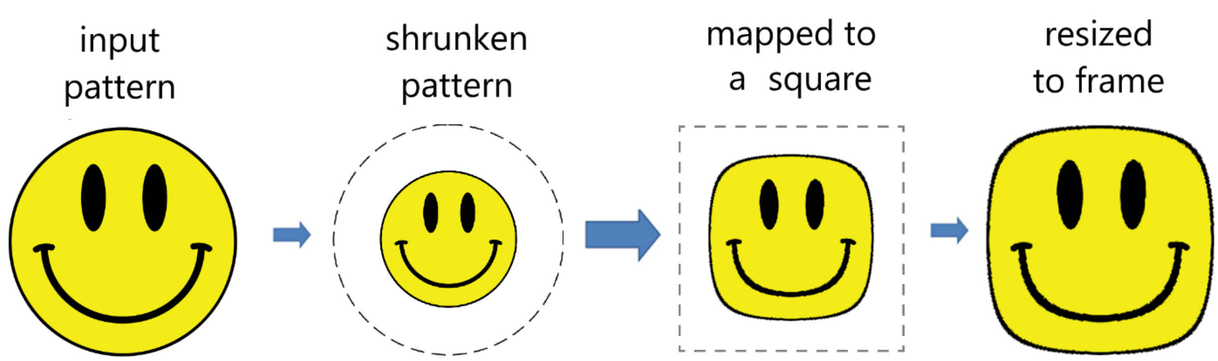

Figure 7: *Pipeline for mapping the disc to a squircle*

There are two uniform scaling constants used in this pipeline. The first one is a shrinking factor. The value of this is the same as the squareness of the desired squircle, i.e., it has a numerical value of q. The second scaling constant is a supplemental enlargement factor needed to resize the squircle at the end of the pipeline. It has a numerical value of $\frac{1}{\Omega(q)}$.

This mapping from the Poincaré disk to the complex squircle is conformal. We will give a brief informal reason for this. First, observe the equation for mapping the Poincaré disk to the conformal square in Figure 5. This equation is conformal because it obeys the Cauchy-Riemann equations in complex analysis. Since we are doing uniform scaling before and after this step in the pipeline, we get this amended mapping equation

$$\hat{\boldsymbol{z}} = \frac{1}{\Omega(q)}[1 - i - \frac{\sqrt{-2i}}{K_e} F(\cos^{-1} q\boldsymbol{w}\sqrt{i}\ \ ,\tfrac{1}{\sqrt{2}})\,]$$

This is the effective equation for mapping the Poincaré disk to a complex squircle of squareness q. This mapping is also conformal because it still satisfies the Cauchy-Riemann conditions. The complex variable $\hat{\boldsymbol{z}}$ represents points inside the complex squircle. The equation can be shortened to $\hat{\boldsymbol{z}} = \frac{\Omega(q\boldsymbol{w})}{\Omega(q)}$.

Squircle Limit Revisited. It is now timely to revisit the "*Squircle Limit*" art piece to provide more details. This artwork is inherently hyperbolic because it starts from a {4,8} tiling of the hyperbolic plane. This is a tiling with 8 hyperbolic squares joining together at each vertex. Afterwards, each square is subdivided into 4 smaller squares and colored to form a hyperbolic checkerboard pattern. The 4 smaller squares are then adjusted and filled-in with fishy details [7]. These steps can be visualized within the conformal square diagrams in Figure 8.

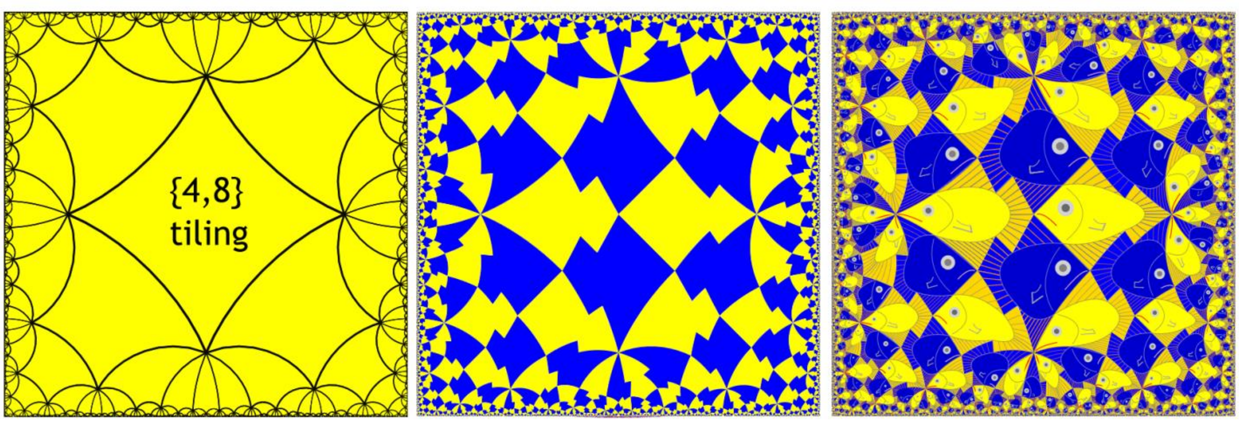

Figure 8: *Geometric scaffolding for interlocking fishes in the conformal square*

Morphing the Poincaré Disk to a Square. The process of morphing in computer animation is simply about being able to interpolate smoothly between two key figures. Since we can map the Poincaré disk to the complex squircle of arbitrary squareness value, we effectively cover the entire spectrum of shapes between the circle and the square. This means that it is possible to use the intermediate squircle mappings to produce a smooth morphing video of the Poincaré disk changing into the conformal square. An example morphing sequence is shown in Figure 9. The pattern consists of an order 7-3 rhombille tiling of the Poincaré disk.

From Circle to Squircle. The pattern inside the circular disc to be mapped to the squircle need not be hyperbolic at all. The pipeline shown in Figure 7 will work for any input pattern that is circular. For example, this pipeline can be used to convert circular corporate logos and university seals into squircles. Moreover, we are not restricted to the conformal method of mapping the disc to a square. In the next section, we will expound on this idea and discuss other artistic possibilities.

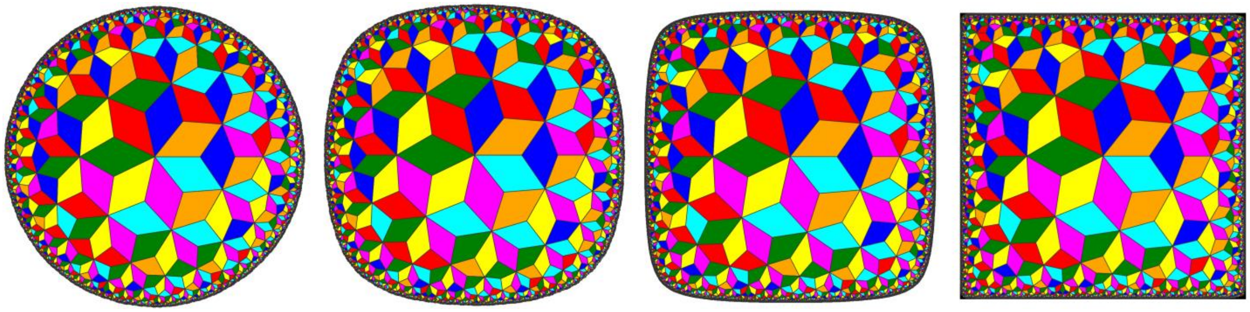

Figure 9: *Morphing the Poincaré disk into the conformal square*

Other Artistic Uses

Although this paper puts emphasis on producing hyperbolic patterns inside squircles, the key ideas introduced in this paper are not limited to such applications. There are many avenues for variation that can be explored further. This section will discuss using these variations for artistic applications.

Other Mappings. Even though this paper primarily focuses on using the conformal square mapping in the circle to squircle pipeline, we are certainly not restricted to only using this mapping. It is possible to use other disc-to-square mappings to morph the circle to a square. We have proposed several alternative mappings in previous Bridges conferences [5,6]. Any of these mappings can be used as a replacement to the conformal square mapping in the pipeline described in Figure 7. For example, the *poor man's disc to square mapping* [6] can produce decent results. However, the intermediate squircles for this mapping are of the Fernandez-Guasti type instead of the complex squircle. Also, the uniform scaling factors for shrinking and enlargement in the pipeline will be different.

From Square to Squircle. This paper does not cover the inverse mapping process and equations, but we would like to mention that the inverse process was thoroughly discussed in previous Bridges conference papers [5,6]. Meanwhile, it is possible to map the square to a squircle if one has an invertible disc-to-square mapping available. The pipeline for this process is shown in Figure 10. Note that this pipeline requires the circle-to-squircle mapping as part of the process. This might seem like a circuitous route but it is completely viable. In fact, it highlights the importance of invertible mappings in practice.

Morphing a Square Diagram to a Circular Disc. Using the square to squircle pipeline just discussed, it is possible to metamorphose square diagrams into circular ones. Two morphing sequences are shown in Figure 11 to illustrate this. The central tenet of this transmogrification is that the squircle can be used in intermediate frames to produce a smooth morphing video.

Adding Eccentricity. Although we will not discuss details in this paper, we would like to mention that there is a simple way [6] to incorporate eccentricity to the mappings thereby making them work with rectangles and ellipses. This extension also applies to the squircles in between. An example morphing sequence with eccentricity is shown at the bottom diagram of Figure 11.

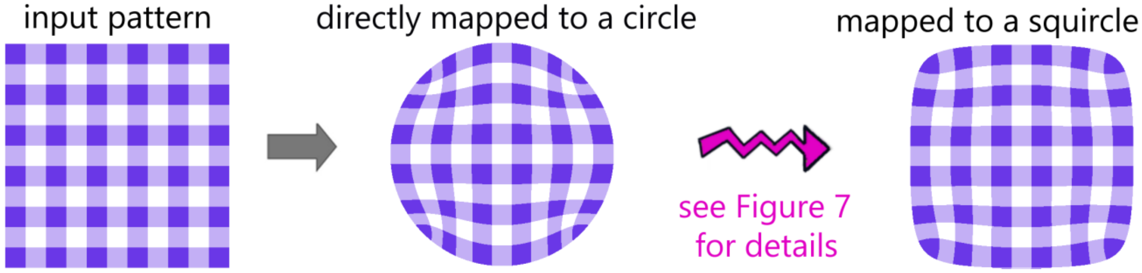

Figure 10: *Pipeline for mapping the square to a squircle*

Figure 11: *Morphing square diagrams into circular discs: the Monopoly™ board game by Hasbro® (top) and the Johannes Kepler University logo (bottom)*

Summary

We presented an art piece named "*Squircle Limit*" as an homage to Escher. We then discussed our methodology for creating this artwork. In addition, we presented a method for algorithmic metamorphosis of the circular disc to the square and vice versa. Many examples of our morphing videos can be viewed in https://squircular.blogspot.com/2019/01/hyperbolization.html

Acknowledgements

We would like to thank the M.C. Escher Company for the use of "*Fishes and Scales*" in Figure 2. We would also like to thank Edward Ashford Lee and Bharath Sriraman.

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