A Fisheye Gyrograph: Taking Spherical Perspective for a Spin

António Bandeira Araújo

DCeT, Universidade Aberta, Lisbon, Portugal; antonio.araujo@uab.pt

Abstract

In this workshop we will draw a total view of our environment in azimuthal equidistant spherical perspective. This is the 360-degree generalization of the so-called fisheye perspective, which was first formalized by Barre and Flocon in the 1960s, to capture a 180-degree view around an axis. In a recent work I have generalized this construction to the 360-degree view using ruler and compass, but in this workshop we will use a new gridding scheme based on a simple mechanism – a gyrograph – that is far more practical for drawing outdoors. We will also briefly discuss how to abandon the grid, to make freehand sketches that follow fisheye perspective rules only qualitatively. We will use custom Geogebra tools to exemplify some of the spherical perspective constructions.

Introduction

In Bridges Stockholm 2018, I presented a workshop on equirectangular spherical perspective [1]. This is a good perspective for wide panoramas and has the advantage that it can be readily visualized as an immersive VR experience [2]. But for some drawings, fisheye perspective is more natural and pleasing to the eye. Historically, it was the first spherical perspective, formalized in its 180-degree form by Barre and Flocon in the 1960s [3]. In a recent work in the Journal of Mathematics and the Arts [4] I extended that formalization to the 360-degree view. This year I propose that we explore this perspective, not according strictly to the method presented in that paper, but through a new gridding scheme that avoids explicit use of the compass and uses instead a simple mechanical gridding device – a *gyrograph*. This makes the method much easier for outdoor use or for quick sketching. I have developed it with *urban sketching* [5] in mind – a specific practice of on-location quick drawing - but it may be useful for other kinds of drawing, both from imagination and from observation, whenever moderate precision is sufficient.

Quick Review of Spherical Perspective – And a New Gridding Scheme

Spherical perspective works thus: choose a point ***O*** for the viewer's eye. Place a sphere around it. Project the 3D objects around the viewer onto the sphere, radially. You get a 2D drawing on the sphere's surface, that, seen from ***O***, looks just like the original 3D scene – a *trompe l'oeil*. We say that the drawing is an *anamorph* of the original scene [6]. The spherical perspective is what we get by flattening the anamorph onto a plane, using a cartographic map. The choice of the map names the specific spherical perspective. Here we use the azimuthal equidistant map to flatten the drawing, hence this perspective carries that name. It is also known colloquially as *fisheye* perspective, when only the anterior hemisphere is drawn.

The azimuthal equidistant map is obtained as in Figure 1: choose points ***F*** (for *Forward*), ***R*(**ight), and ***U***(p) on the sphere, arbitrary but for the requirement that $(\overrightarrow{OF}, \overrightarrow{OR}, \overrightarrow{OU})$ is a right-handed orthonormal referential. Mark points ***B***(ack), ***L***(eft), ***D***(own), antipodal to ***F***, ***R***, ***U*** respectively. Pierce the sphere on ***B***, release its meridians there, and straighten them onto the tangent plane of its antipodal point ***F***, in such a way as to preserve both the lengths along each meridian separately and the angles of the meridians at ***F***. This flattens the sphere onto a disc of center ***F***, isomorphic to the sphere except at point ***B***, that becomes the outer circle of the disc. The total disc is composed of an inner disc (with half its radius) representing everything in front of the observer (the *anterior view*) and an outer ring representing everything to its back (*posterior view*). The circle (*equator*) separating these zones represents the *observer's plane* ***UOR***, passing through $\boldsymbol{O}$ orthogonally to $\overrightarrow{\boldsymbol{OF}}$. The anamorphic drawing on the sphere becomes a flat drawing on this disc, and this flat drawing is what we call the *spherical perspective*.

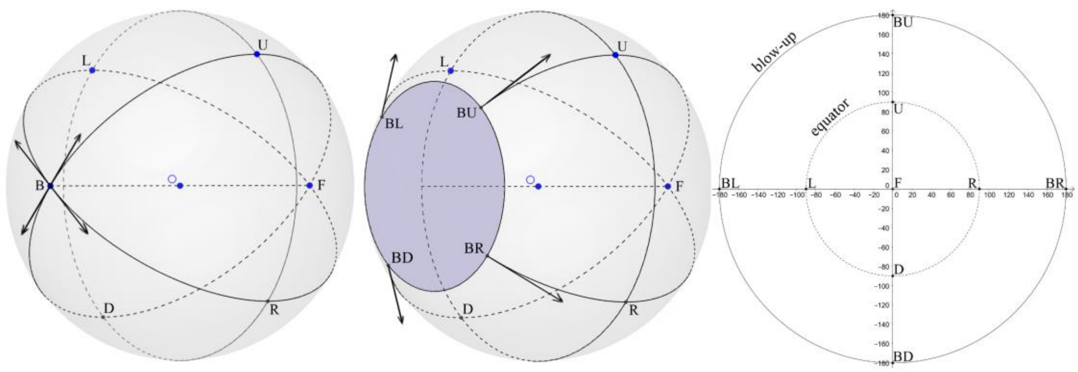

Figure 1: *Azimuthal equidistant flattening of the sphere: Set loose the meridians at B and straighten them radially at F to get a disc. This disc can be divided into an inner disc with half its radius, that represents the anterior view, and an outer ring representing the posterior view.*

The way to draw such a perspective in ruler and compass can be described in a couple of rules, (although the consequences of those rules take some lengthy explaining). Barre and Flocon's work in [3] can be summed up thus: 1) images of spatial lines in the anterior view are well approximated by arcs of circles. 2) Lines are of two types: either they cross the plane of the observer or they don't. In each case you plot an arc of a circle through three special points. In the first case these points are an interior point of the line, where it crosses the plane of the observer, the antipode of that point, and a vanishing point. In the second case you find two vanishing points on the observer's plane and an interior point in the inner disc.

In the same way, my own work in [4] can be described very quickly as a way to piggyback on Barre and Flocon's method so as to draw the posterior view, by relying on two principles: a) Given a point P on the perspective disc, its antipode is the point $P^{\star}$ on ray $\overrightarrow{PF}$ such that $|P^{\star}P|$ equals the radius of the disc. b) To plot a line on the posterior view, find three adequate points on the line (again there are two cases), find their antipodes on the anterior view, plot the anterior line through these using Barre and Flocon's method (this is the piggyback step), then take antipodes again to get the posterior line. Just as in Barre and Flocon, this gets longer in the exposition, but the principles are rather simple.

In order to plot the perspective images of spatial lines, note that each line defines a single plane through $\boldsymbol{O}$, which in turn defines a geodesic of the sphere. The image of the line is half the image of the geodesic (a meridian), ending at two mutually antipodal (i.e., diametrically opposite) vanishing points. Hence if we can plot all geodesics (i.e., images of planes through $\boldsymbol{O}$) we can plot the shapes of all possible line images.

Note that in both methods above we have to find three special points for each line. We cannot simply find (through ruler and compass) the image of the line segment joining two arbitrary points. This turns out to be rather awkward sometimes. There is however a graphical way of finding such a line.

Consider the *plane of the horizon*, *H*=***LOF***. Call *core family* of geodesics to those arising from the planes π_α that go through $\boldsymbol{O}$, $\boldsymbol{L}$ and $\boldsymbol{R}$, and such that $\angle(\pi_\alpha, H) \in [0, \pi/2]$. If we plot these at 5-degree intervals we get the curves in Figure 2 (left). Note that each geodesic is a smooth, closed curve. I have plotted them in different colors, so you can follow them easily as they cross the equator. Their anterior parts (in the inner circle) look like arcs of circles but their posterior halves are quite oddly shaped. In [4] I have shown how to plot them. The core family, itself not forming a grid, is a grid generator. Join it with its rotations by 90, 180, and 270 degrees and you get the grid of vertical and horizontal planes which is often used for gridding methods; but such a grid is not only very "busy", with crisscrossing lines that confuse the sketcher's eye, but is missing all diagonals, hence has limited scope for constructions.

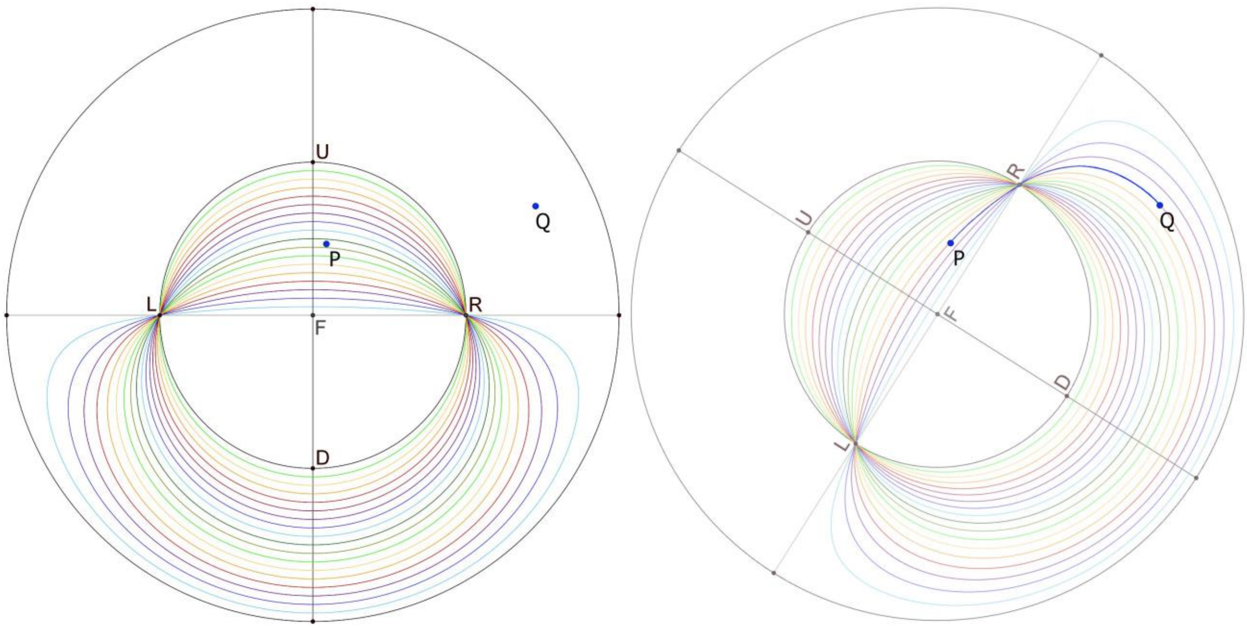

Figure 2. *Left: Geogebra plot of the geodesics through L and R, in a range [0,90º] for the dihedral angle with the horizon plane, and spaced by 5º intervals. Note the two arbitrary points P and Q. Right: All other geodesics can be found from these by rotation around F. Given two points P and Q, a rotation around F finds the only geodesic that joins them. This gives us the correct shape of curve PQ, the perspective image of the spatial line segment **PQ**, by just tracing over the found line.*

However, rotating the planes of the core *freely* around ***OF***, all planes through ***O*** can be obtained. But this spatial rotation corresponds to a rotation of the perspective disc around F, and every line image is contained (is half of) an image of one such plane. Hence, given arbitrary points P, Q on the perspective drawing, images of spatial points ***P*** and ***Q***, you can spin the grid of geodesics around F and you will find that one and only one of those curves goes through P and Q (approximately, of course, since we are only plotting at 5º intervals), and that curve is guaranteed to be the correct perspective image of the spatial segment ***PQ***.

This workshop will explore this fact in order to draw fisheye perspective lines very easily. We build a simple drawing device (let me whimsically call it a *gyrograph*): through a hard base put a flathead nail, point up, that pierces through F a plot of the core family. On top of this plot let the nail also pierce a piece of tracing paper, on which we will draw (and then, for safety, cap the point). The two sheets are stuck together by the nail fixed at F, but the tracing paper can freely spin on top of the plot, to easily find and trace any line through given points. This simple device is not a grid – the plotted lines do not cross - but it implements a *gridding scheme* and carries implicitly all possible fisheye grids. It is a very convenient set-up, much more portable than ruler and compass for the artist interested in quick or outdoor sketching.

Plan of the Workshop

The workshop will consist of a brief theoretical introduction followed by drawing practice. Facility in drawing is a plus but not a requirement. The number of participants should not exceed twenty. All materials (pencils, paper, grids, etc.) will be provided by the instructor. We will proceed thus:

1) We will briefly introduce the azimuthal equidistant flattening. We will discuss how interior and vanishing points of a spatial line are found in the anterior half-space, following Barre and Flocon. We will show how to plot horizontals and verticals, and how to find an arbitrary point through their intersection.

2) We will briefly describe the 360º perspective of [4], pointing out that in it every line has exactly two vanishing points. We will explain the construction of the geodesic guidelines in Figure 2 (left). We shall illustrate these constructions in Geogebra. This will conclude the exposition section.

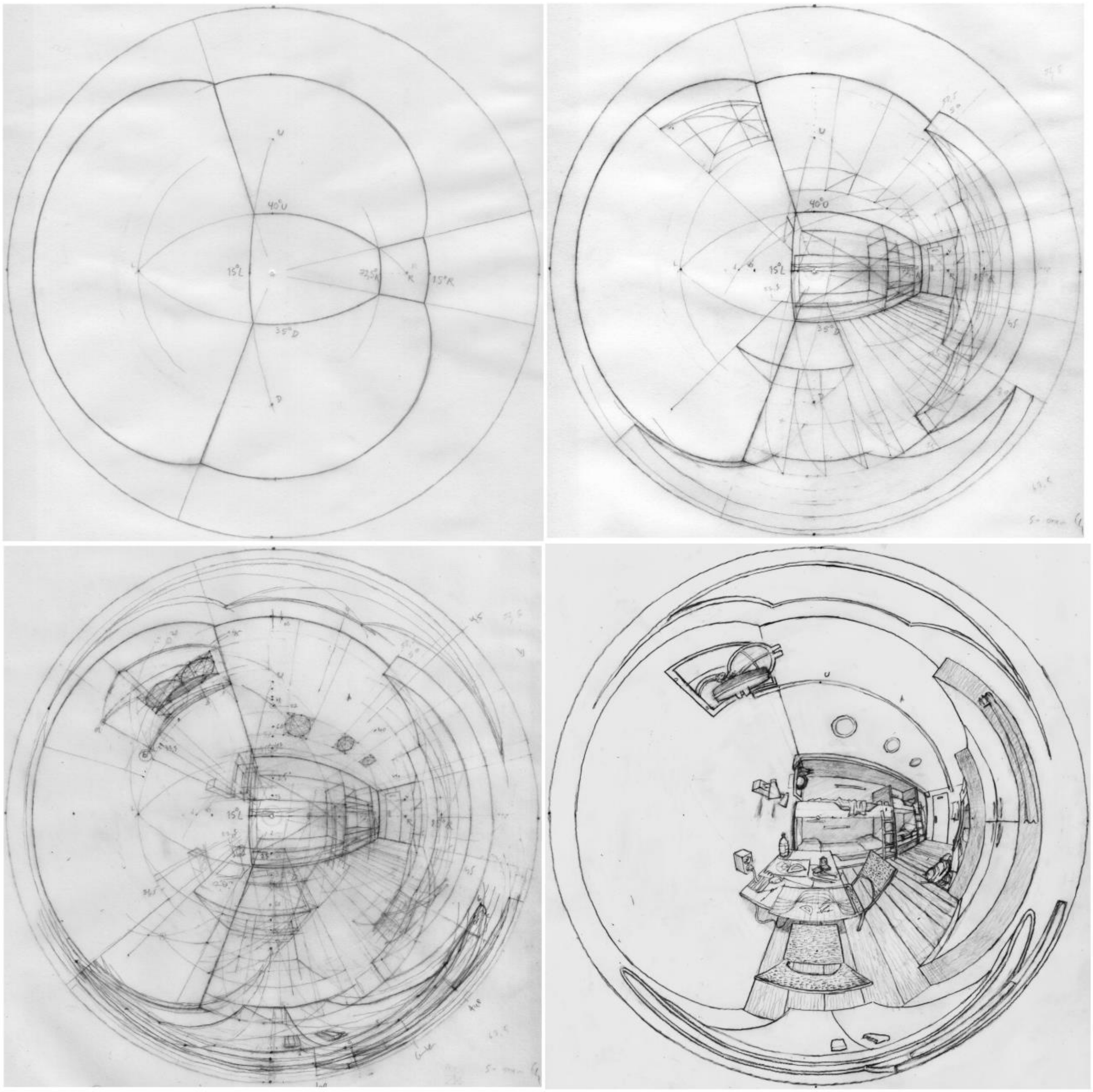

Figure 3. *Steps in the drawing of a room (ship cabin), from basic lines to final rendering. Notice the subdivision and multiplication of segments, and the use of vanishing points to obtain repeating patterns, as in the floor planks (a variation on the classical problem of uniform tiling).*

Now we are ready to draw. Gyrographs will be distributed to the students along with pencils and tracing paper on which they will draw. The students need not bring any material.

3) We will see how to measure points from observation, through the astronomer's trick: a closed fist held flexed at 90º at the end of a stretched arm subtends approximately 10º (some calibration required for each body, of course). We will show how to use this and the gyrographs to plot horizontals and verticals, and how to find through rotation the segment joining two arbitrary points.

4) Now we draw boxes! Namely, the basic lines of the room we are in (Figure 3 – Upper left). We do this by identifying two key points on each edge of the room, one interior point and one vanishing point. We will see that most edges in the room are redundant. In fact, only five edges need be measured to fully determine the perspective of a box. This will speed up the drawing considerably.

5) Next we do perspective arithmetic! We will learn how to subdivide a box by drawing the diagonal segments between the vertices of each rectangular face. Where these diagonals cross we find the center of the face. This is easy through our gridding scheme: just rotate the gyrograph until you find the lines through the vertices. Similarly, we will perform multiplication of segments (Figure 3). In this way a found measurement may be replicated at will.

6) We learn to draw repeating structures through the use of the vanishing points of their diagonals. In particular we learn how to construct uniform grids, a classic problem of a perspective. This is useful for tiled floors, but also for the floor planks in Figure 3 or the steps in Figure 4.

7) We learn how to do inclined planes. If we don't have one in the room we make it up. This would be good for stairs, among other uses (Figure 4), but there won't be time for all that in this workshop, just the ramp. In any case the principle is the same, relying on setting up the adequate vanishing points.

8) Once we have enough construction lines we can just eyeball the small objects in the room and draw freely the various objects we observe into the places dictated by the geometric setup.

9) Finally we will discuss how to break the rules: how to draw intuitively, taking into account what we know about vanishing points in spherical perspective and following the rules only where we please, with a view to a pleasing rather than strictly correct effect. In particular, we often will not want a full 360 view, and we are free to crop according to convenience (Figure 5).

Throughout all this, instruction will be facilitated by Geogebra illustrations with the instructor's custom Geogebra tools. These will be useful to project instructions on the screen for the participants, but also will provide them with an optional tool to clean up drawings when vectorial formats are required. The tools will be made available at the instructor's website.

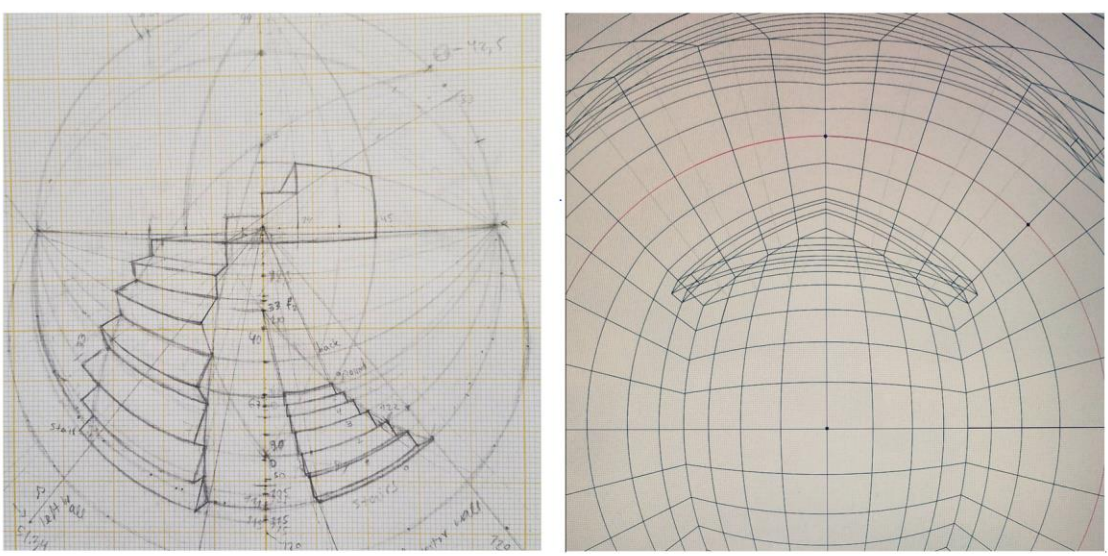

Figure 4: *Inclined planes. Left: Setting up the vanishing points for flights of stairs. Right: Geogebra construction of the inclined roof of a house in fisheye spherical perspective.*

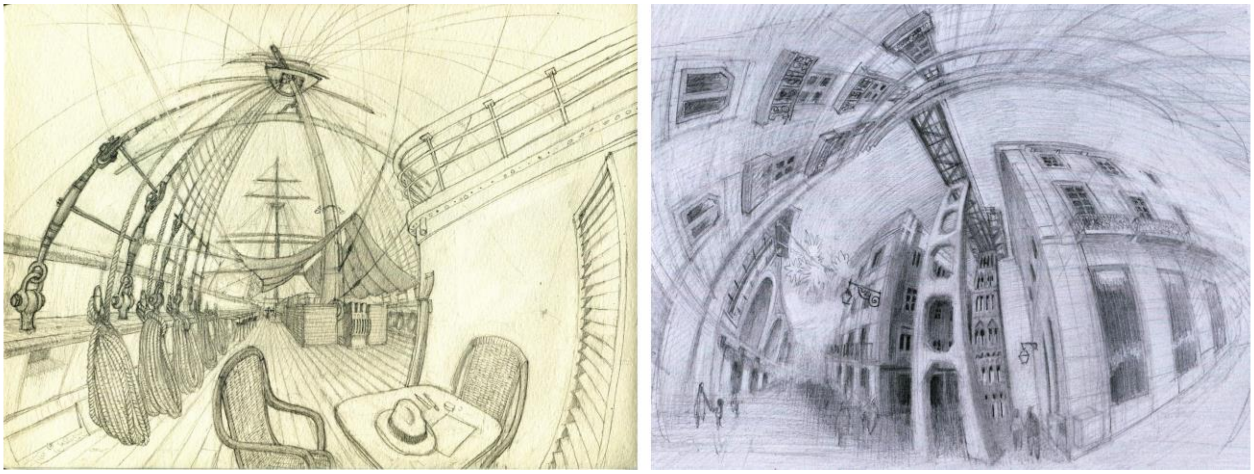

Figure 5: *Cropped Spherical Perspectives. Left: Santa Justa Elevator seen from Chiado, in Lisbon, Portugal. Right: Deck of the af Chapman, moored in Stockholm, Sweden.*

Summary and Conclusions

A recent paper by the present author has shown how to solve the azimuthal equidistant spherical perspective using ruler and compass constructions. Here we present a new gridding scheme that simplifies these constructions and provides a method adequate for drawing in the field, where ruler and compass would be cumbersome. Further material will be made available at the author's website [7], including printable grids and Geogebra tools.

Acknowledgments

The author was a member of the research centers CMAF-CIO and CIAC-UAb during the development of this work, and was financially supported by Portuguese national funds through FCT- Fundação para a Ciência e a Tecnologia, within projects UID/MAT/04561/2013 and UID/Multi/04019/2013.

References

[1] A. B. Araújo. "Let's Sketch in 360°: Spherical Perspectives for Virtual Reality Panoramas." *Bridges Conference Proceedings,* Stockholm, Sweden, Jul. 25-29, 2018, pp. 637-644. http://archive.bridgesmathart.org/2018/bridges2018-637.pdf

[2] A. B. Araújo. "Drawing Equirectangular VR Panoramas with Ruler, Compass, and Protractor." *Journal Of Science And Technology Of The Arts,* vol. 10, no. 1, 2018, pp. 15-27. DOI:10.7559/citarj.v10i1.471

[3] A. Barre and A. Flocon. *La perspective Curviligne.* Paris: Flammarion, 1964.

[4] A. B. Araújo. "Ruler, compass, and nail: constructing a total spherical perspective." *Journal of Mathematics and the Arts,* vol. 12, no. 2-3, 2018, pp. 144-169. DOI: 10.1080/17513472.2018.1469378

[5] Urban Sketchers Blog. http://www.urbansketchers.org/.

[6] A. Araújo. "Anamorphosis: Optical Games with Perspective's Playful Parent." *Proceedings of the Recreational Mathematics Colloquium V G4G,* Lisbon, Portugal, 2017, pp. 71-86. https://repositorioaberto.uab.pt/handle/10400.2/6647

[7] Notes and materials on spherical perspective. http://www.univ-ab.pt/~aaraujo/full360.html.