Building Polyhedra Models for Mathematical Art Projects and Teaching Geometry

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Abstract

We demonstrate a simple way of building models for a large family of polyhedra. In the workshop, we will build such models using special precut shapes, but the method can be used with many common materials (wood, paper, plastic, wire, etc.) or even recycled objects, making it very suitable for a mathematical art project and/or a class activity.

Introduction

Many works of geometric art are based on polyhedra models and building such models is a favorite activity in a math or art class. In this workshop we will build, in a modular fashion, models of many polyhedra. The constructs are not exact models, but they still capture all polyhedra elements (vertices, faces, edges) and their connectivity. The model components can be elongated shapes made of a flexible material such as pieces of wire, strips of paper or plastic, etc. These flexible components are under tension and the models resemble spherical polyhedra. The crux of the building method presented here is a simple rule for connecting these parts. To save time when building the models, we will provide precut paper and/or plastic shapes and the participants will be able to take home the models they built.

We will also discuss some variations of the connection method that extend the range of possible model components to practically any elongated shape (including rigid ones) and produce two other styles of models, intertwined loops and tensegrity. This model building technique was first presented in [1] and the interested reader is referred to this work for mathematical details and proofs. A related approach is described in [2].

Model Construction

The building blocks for the constructions described here are long pieces connected using the following simple rules:

1. each piece end is connected to the middle of another piece, and
2. each piece middle is connected to exactly two ends.

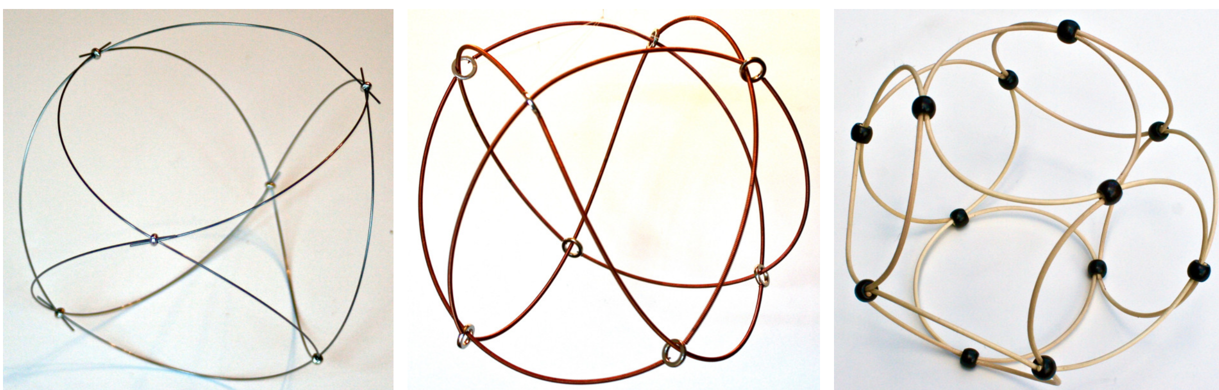
The construction process must allow a range of angles between the connected pieces; this is achieved through the flexibility of the pieces themselves and/or flexibility in the connection mechanism. Figure 1 illustrates this modeling technique using plastic strips connected with ties, while Figure 2 shows models build with other materials.

The following results about models built in this manner were presented in [1]; the interested reader should refer to that work for further details. By "polyhedron" we mean here one whose surface is a topological sphere. Let $\mathcal{P}_4$ be the class of polyhedra where all vertices have degree 4.

- We can build, with the above rules, a model of any polyhedron $P \in \mathcal{P}_4$. If P has n vertices (and thus $2n$ edges), the model will have n pieces, with each piece centered on a vertex and corresponding to two non-consecutive (opposite) edges adjacent to this vertex. The polyhedron faces are in one-to-one correspondence with the model "faces" (regions bordered by consecutive half-pieces).

(a) *Two ends to a middle connection* **(b)** *Six piece octahedron model with 8 triangular "faces"* **(c)** *Other models*

Figure 1: *Models built with plastic strips strips and ties*



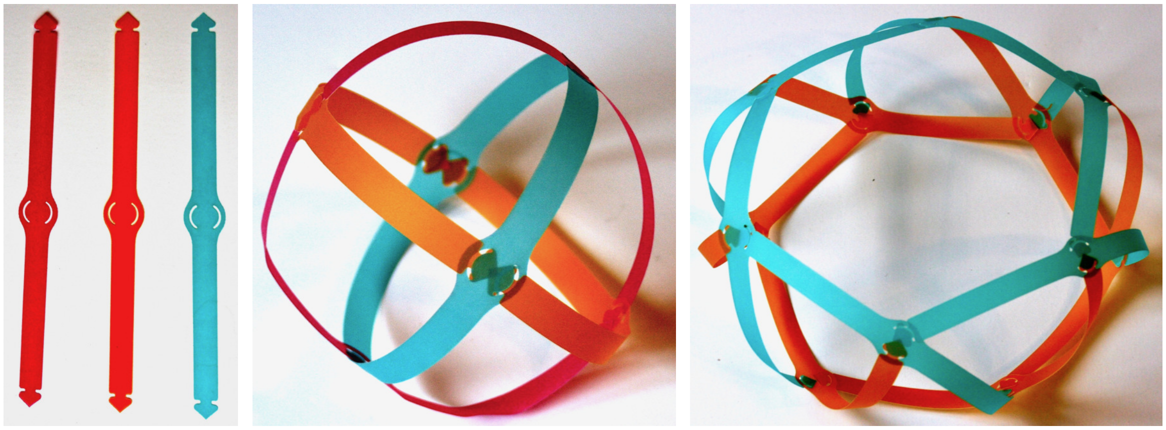
(a) *Octahedron, steel wire and beads* **(b)** *Square antiprism, piano wire and bead frames* **(c)** *Cuboctahedron, reed and beads*

Figure 2: *Models built with various materials. No glue was used, the tension created by the pieces makes the connecting beads stay in place.*

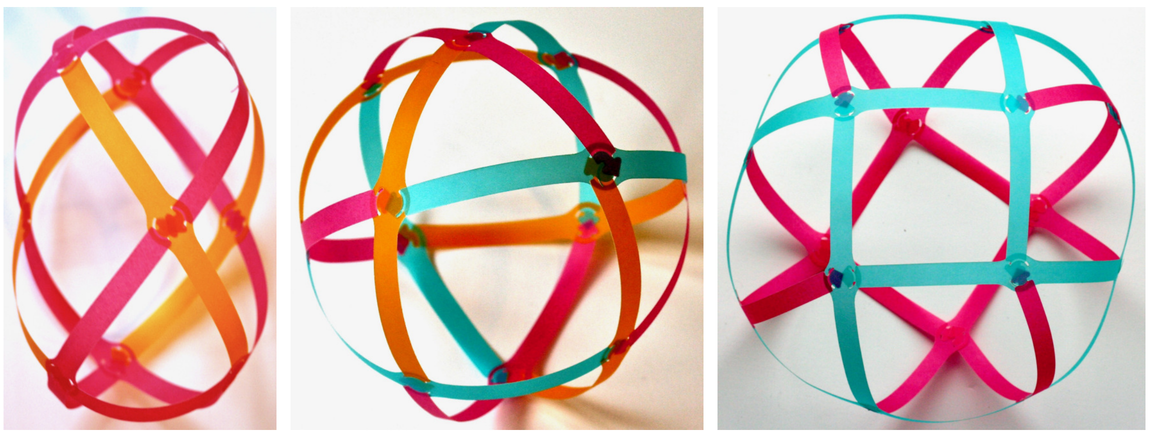
- For any polyhedron P with n edges, its *rectification* (*ambo* in Conway's polyhedron notation) aP has n vertices and is in $\mathcal{P}_4$; it thus has a model with n pieces. The pieces correspond to the edges of P and are connected the same way as these edges.[1] P', the dual of P, will generate the same model since $aP' = aP$; the model faces correspond to the faces of both P and P'.
- There are models of (conventional, flat-faced) polyhedra with n pieces for $n = 6$ and any $n \geq 8$.[2] In fact, we can add a new piece to any model with n pieces, $n \geq 8$, to obtain a model with $n + 1$ pieces.

[1]By definition, two polyhedra edges are connected if they share both a vertex and a face.

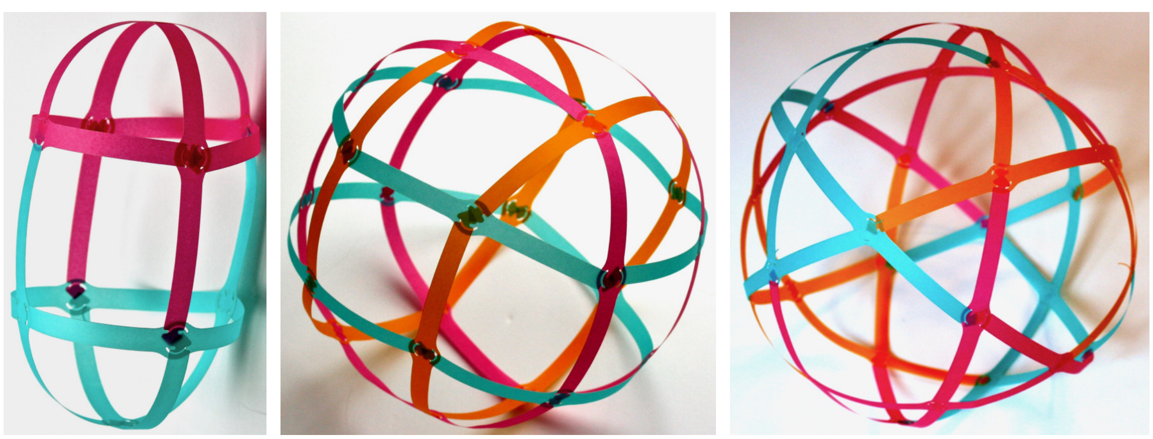
[2]Because there are polyhedra with n edges for each such n. There are also models of polyhedra that only exists in spherical form, including some that have fewer than 6 pieces, see Figure 4.



(a) *Precut pieces* **(b)** *Octahedron (6 pieces)* **(c)** *Hexagonal antiprism (12 pieces)*



(d) *Rectified triangular prism, aP_3 (9 pieces)* **(e)** *Cuboctahedron (12 pieces)* **(f)** *Square gyrobicupola, J_{29} (16 pieces)*



(g) *Elongated square bipyramid, J_{15} (10 pieces)* **(h)** *Rhombicuboctahedron (24 pieces)* **(i)** *Icosidodecahedron (30 pieces)*

Figure 3: *Precut pieces and some models for the workshop activity*

(a) *4-hosohedron (2 pieces)* **(b)** *3-lucanicohedron (3 pieces)* **(c)** *4-lucanicohedron, rectification of (a) (4 pieces)*

Figure 4: *Models of some spherical polyhedra built with the precut pieces used in the workshop activity*

Activity

The workshop activity will consist in building various polyhedra models using precut paper and/or plastic pieces as shown in Figures 3 and 4. We will provide enough pieces and the participants will be able to take home the models they build. The precut pieces snap together easily and allow us to focus on building the models and exploring their properties.

Note that the pieces in Figure 3 (a) are used just for convenience; the construction method demonstrated in the workshop can easily be adapted to other shapes and materials. This model building technique can be used to make many decorative objects (lamps, tree decorations, etc.) or other mathematical art sculptures. In a geometry class, this method can be used as a teaching aid in the study of polyhedra, illustrating concepts such as polyhedral graphs, rectification, duality, etc.

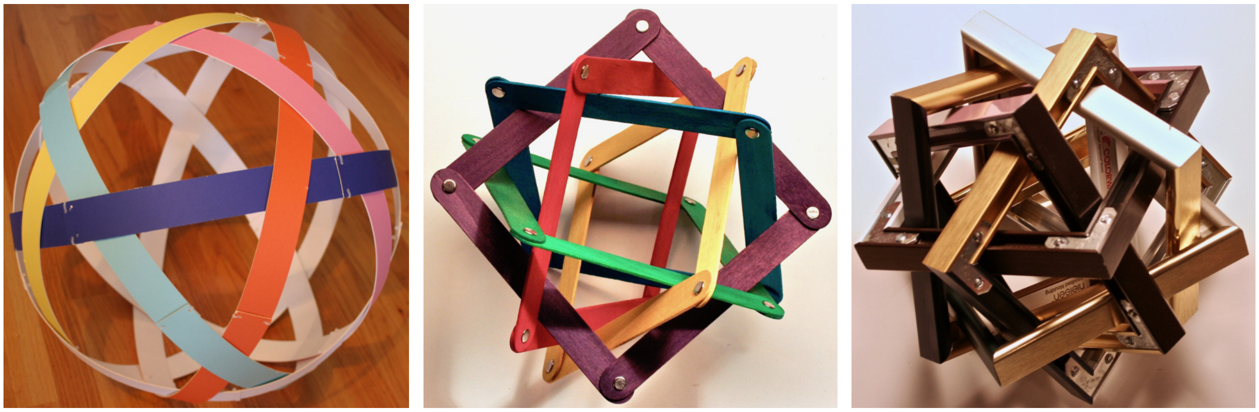
The construction process allows some variations which are described in the next section and will be presented in the workshop.

Variations

There are a number of model building variations that allow the use of more shapes and materials, can produce objects of greater artistic interest, and can also be used to gain a better understanding of some geometric concepts such as duality.

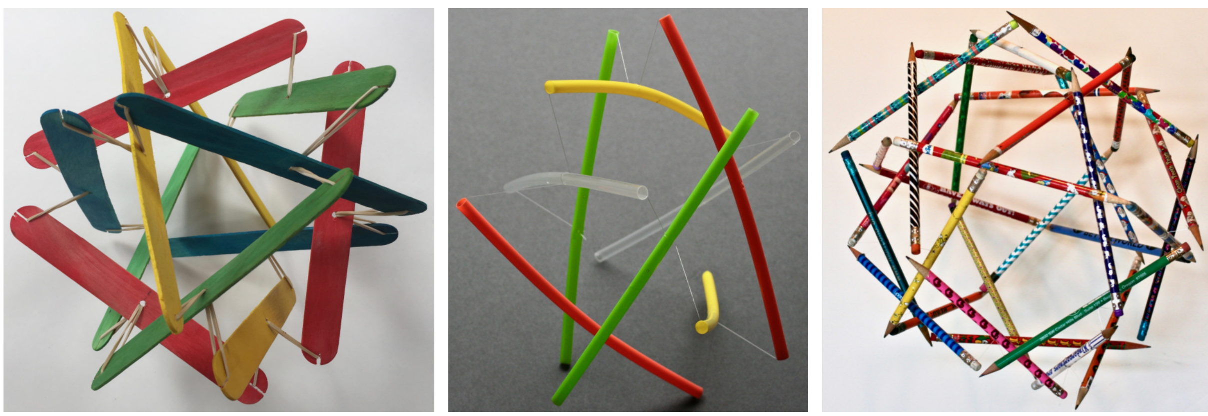
If, in a model, we ignore the piece middles and follow just the connections of piece ends, we can see that the pieces form one or more closed circuits or loops (e.g., three 2-piece, color-coded loops in Figure1 (b)). If the two ends that connect to the same middle are instead connected to each other and the connections are done in a uniform manner, i.e., a piece middle is always below (or always above) the two connected ends, the model becomes an aggregate of perfectly *intertwined loops*. This is apparent in Figure 5 (a) where the 6 5-piece loops (which correspond to the 6 equatorial decagons of the icosidodecahedron) can move with respect to each other, but the model still keeps its shape due to the interweaving. Intertwined loops models can also be built with rigid bars, see Figure 5 (b) and (c).

Another style of models can be built if we keep the rigid bars (instead of the flexible pieces), but revert



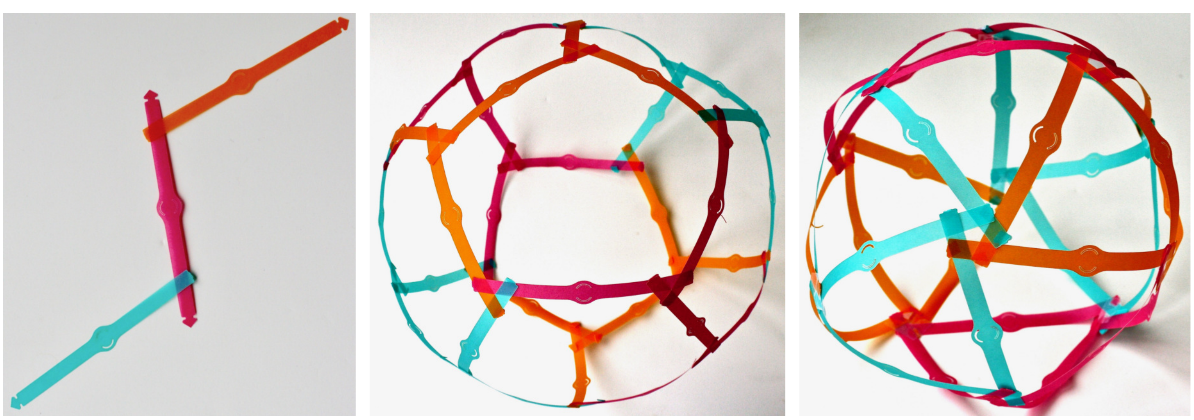
(a) *Icosidodecahedron (30 paper strips, 6 loops)* **(b)** *Rectified gyroelongated square pyramid, a J_{10} (20 craft sticks, 5 loops)* **(c)** *Rhombicuboctahedron (24 frame pieces, 6 loops)*

Figure 5: *Intertwined loops models*



(a) *Cuboctahedron, rubber bands* **(b)** *Square antiprism, a single string* **(c)** *Icosidodecahedron, strings*

Figure 6: *Tensegrity models*



(a) *Split the middle connection point* **(b)** *Dodecahedron* **(c)** *Icosahedron*

Figure 7: *Variations of the 30 piece icosidodecahedron model*

to the "two ends to a middle" type of connections. Since the two ends and the middle they must connect to cannot be brought together anymore, the connections must be made either with strings (for a more "artistic" effect) or rubber bands (easier to tie in a classroom setting). The model becomes now a *tensegrity* construct, see Figure 6. These models are harder to build since all the connecting strings must be tied and adjusted separately. As an exception, the square antiprism model in Figure 6 (b) was built using a single loop of string that threads each straw twice, once along the straw and once through its middle. This is possible because the pieces form a single loop (this is true for the model of a k-antiprism whenever k is not a multiple of 3).

A modification of the connection method can be used with any of the model building variants described above to change the model appearance and make it resemble more closely the modeled polyhedron. As explained above, we cannot directly model, for example, a dodecahedron since the dodecahedron is not in $\mathcal{P}_4$. The closest we can come is to model a icosidodecahedron, the rectification of the dodecahedron; see, for example, Figure 3 (h). In addition to the 12 pentagonal dodecahedron faces, this model has 20 triangular faces corresponding to the faces of the icosahedron, the dual of the dodecahedron. A modification of the connection method that consists in splitting the middle connection point of each piece in two can change the relative size of these faces. If, after the split, we move the two new connection points away from each other, towards the ends of the piece, we can, depending on the direction of the move, make the model resemble either a dodecahedron (by making the pentagonal faces larger and the triangular ones smaller) or an icosahedron (by making the triangular faces larger and the pentagonal ones smaller), see Figure 7.

In general, this technique can be used to bring a model of the rectification of a polyhedron P and its dual P' closer to either P or P'. At the limit, when the small faces degenerate into vertices, the model becomes a wireframe model of P or P' with each model piece corresponding either to an edge e of P or its dual edge e' of P'. This neatly illustrates the fact that the polyhedral graph of $aP = aP'$ and the connectivity graphs of the edges of P and P' are isomorphic.

Summary and Conclusions

We demonstrate a simple method for building models for a large class of polyhedra. The models are built out of pieces made of diverse materials connected in a uniform manner. The method is particularly well suited to upcycling, i.e., creating artworks from reused and recycled objects and materials, see for example Figure 2 (b), Figure 5 (c) and Figure 6 (b), (c).

The workshop's activity can be easily adapted to middle school (or even younger) students. In a math class the models can be used to introduce 2D and 3D geometry facts, counting methods, symmetry, etc, while in an art class the method can be used to make a variety of geometric art objects. The students can start with precut shapes and then move on to other model types; most accessible are the ones illustrated in Figure 1 Figure 5 (a), (b), and Figure 6 (a).

References

[1] Mircea Draghicescu. "A General Method for Building Topological Models of Polyhedra". In: *Proceedings of Bridges 2017: Mathematics, Art, Music, Architecture, Education, Culture*. Ed. by David Swart, Carlo H. Séquin, and Kristóf Fenyvesi. Available online at http://archive.bridgesmathart.org/2017/bridges2017-175.pdf. Phoenix, Arizona: Tessellations Publishing, 2017, pp. 175–182. ISBN: 978-1-938664-22-9.

[2] David Reimann. "Transforming Squares to Strips in Expanded Polyhedral Forms". In: *Proceedings of Bridges 2017: Mathematics, Art, Music, Architecture, Education, Culture*. Ed. by Carlo H. Séquin David Swart and Kristóf Fenyvesi. Available online at http://archive.bridgesmathart.org/2017/bridges2017-435.pdf. Phoenix, Arizona: Tessellations Publishing, 2017, pp. 435–438. ISBN: 978-1-938664-22-9.