Tessellated Seven-Color Tori

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Abstract

The seven-color torus map may be thought of as an artistic form, like haiku or the sonnet in poetry, with specific constraints within which there is artistic freedom. Many interesting variations on the map have been created, but they typically have stayed confined to representing "countries" with very simple polygonal tiles or abstract shapes. We explore the genre further with a design approach using Escher-like tessellations that produce seven-color torus maps suitable for printing on fabric and sewing into toroidal scarves.

Introduction

One particularly entertaining—and certainly colorful—topic in the field of mathematics is map coloring. For those who like math problems with a visual component, it has historically offered up a wealth of imagery compelling enough to populate the internet and adorn the covers of multiple mathematics texts [7,12]. An image search on "mathematical map coloring," "four color theorem," or "seven-color torus" yields an abundance of results in a variety of media, including geopolitical maps, geometric abstractions, and more freeform compositions, both elaborate and simple. However, the maps themselves often do not take full advantage of the possibilities from an artistic or aesthetic standpoint. We are interested in the particular case of the 7-color torus map.

A canonical 7-color torus map, which demonstrates the existence of a map on a torus that requires seven colors (in order that no two adjacent "countries" are painted the same color), is simplicity itself. With just seven "countries" that all border one another, it is a simple matter to check that there is no way to paint it without using a unique color for each of the seven countries (in contrast to planar maps, which are known to require no more than four colors [2]). However, this simple elegance of the map conceptually need not translate into artistic limitations. We present a way to create your own 7-color torus maps using an approach in which the artist can focus on designing country shapes without worrying about disrupting the map adjacencies.

One way to achieve simplicity and understandability in a map is to have countries that are all identical in shape. Accordingly, many of the most readily understood existing maps are composed of seven identically shaped polygons [3,6,7,10]. For example, Figure 1b shows a polygonal 7-color torus map composed of parallelogram-shaped countries, created by Ellie in 2015 to use as a toroidal scarf design. To create the scarf, take the rectangular panel in Figure 1b and identify (or sew together in this case) first the top and bottom, then the left and right, edges of the rectangle. This creates a 7-color mapping on a torus. Figure 1a shows a planar tessellation composed of parallelograms, created by adjoining copies of the map in Figure 1b. Tiling the map this way (where the sewn edges connect) makes it clear how the seven parallelograms of a single panel will meet when it is sewn into a toroidal scarf.

(a) (b)

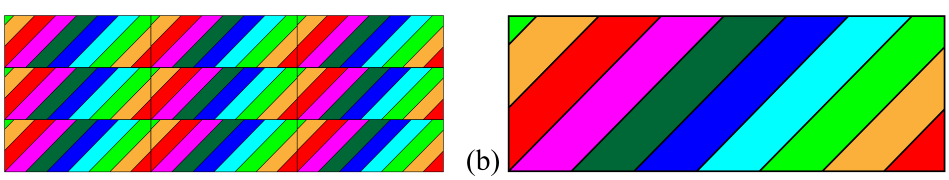


Figure 1: *(a) Nine adjoining copies of the 7-color torus map in (b).*

Observe that at every vertex of the tessellation in Figure 1a, three parallelogram tiles meet, and every parallelogram contains six vertices of the tessellation. So topologically, this is a tessellation of hexagons and the hexagons have the property that opposite edges are congruent by translation (the accentuated vertices shown by red dots in Figure 5a may make this easier to see). From this observation, it's a short step to consider the potential for modifying the country shapes by playing with the tessellation. Once we set up the vertices of a valid 7-color torus map tessellation, we can tinker with the edge contours quite freely—without changing the adjacencies—as long as we keep the tile vertices fixed. For this, we will use *TesselManiac!* [11], a software tool developed by Kevin for creating Escher-like tessellations.

Hexagonal 7-Color Torus Maps

TesselManiac! provides the ability to create tessellations with exactly the properties we need to produce valid 7-color torus maps. To help clarify these properties, let's first look at another polygonal 7-color torus map, this time one made from *regular* hexagons. In a recent article, Moira Chas provides sewing instructions for a 7-color torus map made from seven crocheted hexagons laid out like those in Figure 2 left. She notes that "…it seems natural to work with hexagons, since each region borders six others," and further observes that her resulting map is "…fairly hard to understand" [6]. However, we noticed that it is equivalent in structure to the invertible infinity scarf described in [4], which can be constructed by sewing together the opposite edges of a hexagon. Figure 2 (middle) shows the scissors congruence of Chas' 7-hexagon layout to a single larger hexagon that, when its opposite edges are identified, produces the same hexagonally-based 7-color torus map. Printed and sewn on silky fabric, it should be much easier to comprehend that the resulting object is a toroidal form with a Möbius-like twist (ideal for a scarf), as well as to inspect and verify the map's adjacency properties. Figure 2 (right) shows two equivalent layouts for such a map cut from a planar tiling of the seven hexagons. The rectangular layout, which is scissors congruent to the hexagonal layout [4], produces the same map by first sewing together its short edges and then giving the resulting cylinder a half twist before sewing together the long edges. Either layout can be used to produce the invertible infinity scarf with a Möbius-like twist described in [4].

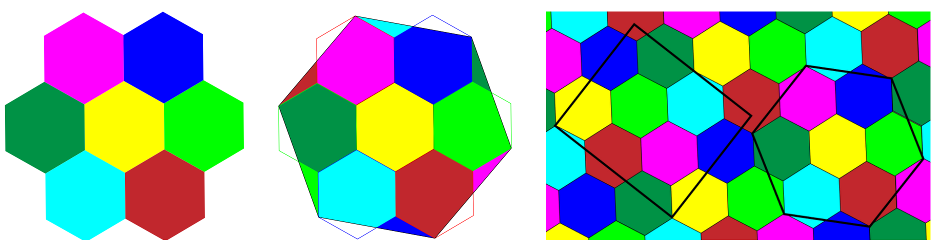
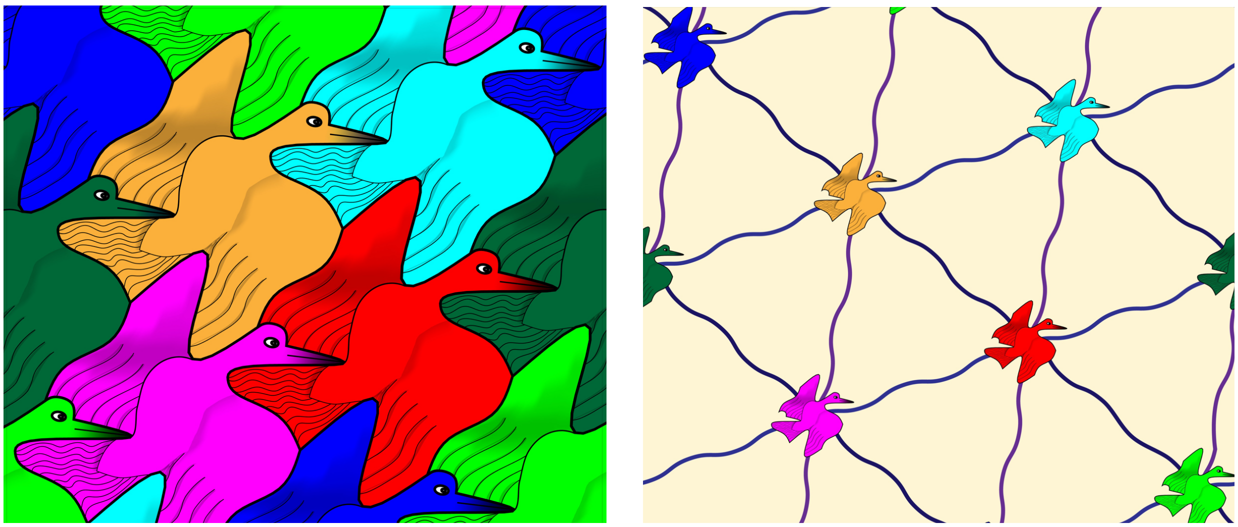


Figure 2: *The seven hexagons in seven colors (left) are scissors congruent to a single large hexagon (middle) and can be cut from a tiling of hexagons in layouts that sew into seven-color torus maps (right).*

Hexagonally-based Tessellations

TesselManiac! provides hexagonal tile types that can be manipulated to construct variations on the hexagonal tessellations shown in Figure 2. Heesch tile Type TTTTTT (Figure 5c), is based on a hexagon whose opposite edges correspond by translations (sometimes called a par-hexagon) [11]. Using it, the 7-color map of birds in Figure 3a was created with a regular hexagon as the starting tile (Figure 5d). The inner details and seven-coloring were added afterwards in *Adobe Illustrator*. Because a tiling based on a regular hexagon produces a pattern suitably proportioned for an *invertible* scarf [4], for fun, we also show an interior fabric (Figure 3b) using the same bird tiles scaled down to form nodes on a fully connected graph. Thus, the final scarf inverts between a 7-color torus map and its dual, the K_7 graph.

Regular hexagons are particularly well suited for designing 7-color torus maps because all edges are congruent and opposite edges are related by translations. They are a special case of a TTTTTT tile and also produce the perfect layout shape needed to make an invertible infinity scarf [4]. But tessellated designs for more traditionally-shaped infinity scarves similar to that in Figure 1 can be created as well. The map in Figure 1 is also an instance of the TTTTTT tile type, only it uses an *irregular* hexagon. In this case, the hexagon has two vertices hidden midway along each short edge of the parallelogram (comparing the red vertices in Figures 5a & b may make this easier to see). Figure 6b shows a tessellation closely related to the parallelogram tessellation in Figure 1 that produces a seven-color torus map scarf with no Möbius twist. It was created using irregular par-hexagons like those in Figures 5a as the starting point, as shown in Figure 5f. Figure 6a shows another TTTTTT design based on irregular par-hexagons that is a stretched-out variant of the birds in Figure 3a (see Figure 5e).



(a) (b)

Figure 3: *(a) 7-color map of birds using a regular hexagon as the base tile for the tessellation. To make it into the toroidal scarf shown in Figure 4, sew together the left and right edges and give the resulting cylinder a half twist before sewing together the cylinder ends (i.e., the top and bottom edges). (b) K_7 graph with no link crossings using scaled down tiles as the nodes, sewn together the same way.*

Figure 4: *Scarf sewn from fabric printed with the map in Figure 3 (displayed in Bridges 2019 online art exhibit [5]).*

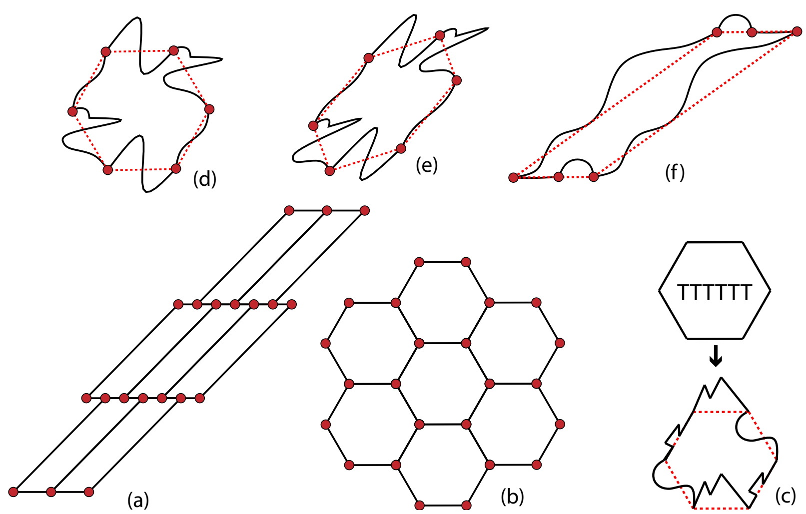


Figure 5: *(a) Irregular hexagon tiling with the hexagons masquerading as parallelograms. (b) Regular hexagon tiling. (c) Heesch type TTTTTT in which opposite hexagon edges correspond by translation. (d,e,f) Hexagonally-based bird tiles.*

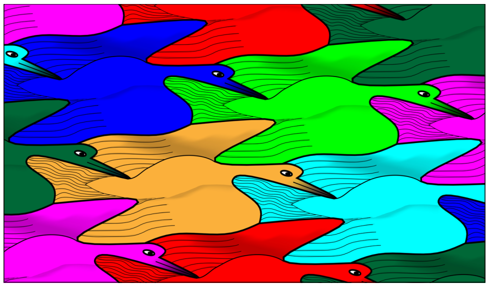
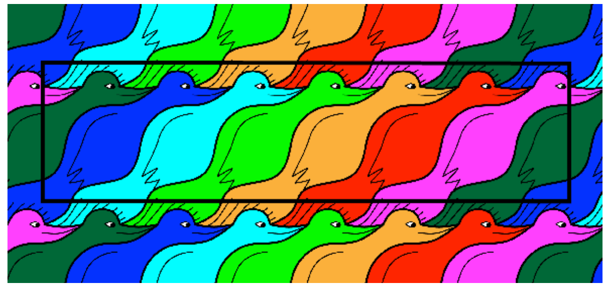

(a)

(b)

Figure 6: *Non-regular hexagonal designs. (a) Top and bottom edges are sewn together first and the resulting cylinder is given a half twist before sewing together its ends. (b)Top and bottom and then left and right (inner rectangle) edges are sewn together. See [1] for hidden seam sewing instructions.*

Summary

Using Heesch tile type TTTTTT (Figure 5c) in which opposite edges of a hexagon correspond by translation, we have created Escher-like 7-color torus maps that can be printed on fabric and sewn into toroidal scarves. We hope others will be inspired to try our approach and use it to further explore the artistic possibilities of the 7-color torus map form.

Acknowledgements

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