The Sum of Odd Integers Quilt

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Abstract

The beauty and symmetry of the image of The Sum of Odd Integers intrigued me. My eye was attracted to the pattern as I imagined using my new Grunge fabric by Moda to sew the quilt. The blue and green fabric with four color values would be very attractive for the four congruent sections of the quilt. I also wanted to highlight the 17 wallpaper patterns in the decorative quilting stitches.

Figure 1: *The Sum of Odd Integers quilt, 58 inches by 58 inches*

The Inspiration

The Sum of Odd Integers quilt was inspired by an image in the book by Roger B. Nelsen: *Proofs Without Words, Exercises in Visual Thinking.* This fascinating book has over 120 samples linking visual thinking with images that represent the mathematics involved. The images/diagrams are not proofs in themselves, but a stepping stone to starting a proof of the concept illustrated. These visual clues stimulate mathematical thought [5]. A total of three images of the sum of odd integers are shown in the book. There are two other books on the same topic of visual proofs authored by Roger B. Nelsen.

The first image in Figure 2 is credited to Nicomachus of Gerasa dating to circa A.D. 100. This image is a very familiar one to mathematicians. The second image in Figure 2 is the pattern that I quilted. There is no source indicated for this image. It is important to note that there are four sums of odd integers summing to $2n^2$. The last image in Figure 2 is a diagram credited to Jenő Lehel.

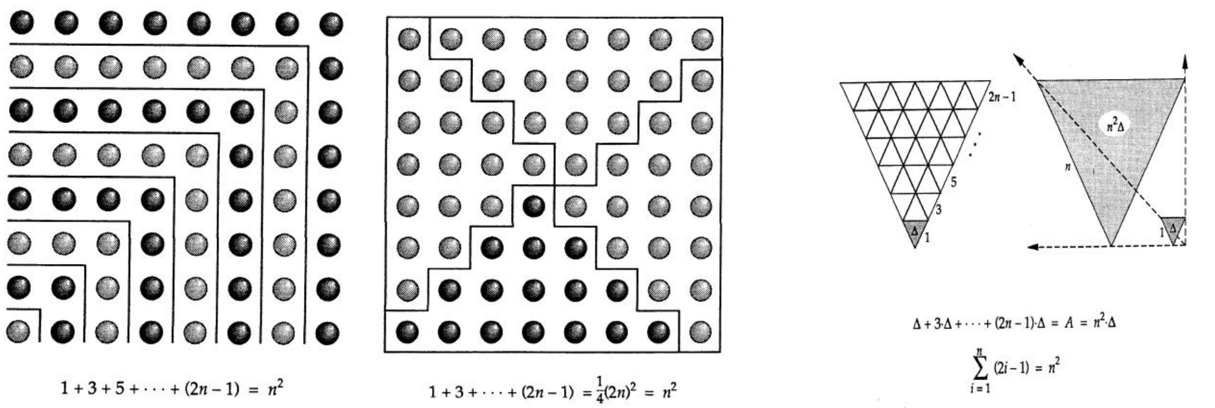

Figure 2: *Three possible images for The Sum of Odd Integers quilt [5]*

Carl Friedrich Gauss

Carl Friedrich Gauss (1777-1855) is recognized as being one of the greatest mathematicians of all time. It is highly likely that his work inspired some of the images in Figure 2. During his lifetime, Gauss made significant contributions to almost every area of mathematics, as well as physics, astronomy and statistics. Like many of the great mathematicians, Gauss showed amazing mathematical skill from an early age, and there are many stories which show how clever he could be [6].

During his primary school years, Gauss' teacher asked the class to add together the numbers from 1 to 100, assuming that the task would occupy the class for quite a while. After a few seconds, Gauss reported the sum of 5050. Gauss showed the class his solution: he added the numbers in pairs. 1 + 100 = 101, 2 + 99 = 101, 3 + 98 = 101, and so on. The total would be 50 sets of 101, which is 5050.

It is remarkable that an elementary school child discovered this method for summing sequences for numbers, but of course, Gauss was a remarkable child. Fortunately his talents were discovered, and he was given the chance to study at the university. By his early twenties, Gauss made discoveries that would shape the future of mathematics [6]. His work may have inspired work on summing other sequences.

This representation of how Gauss solved the problem of summing integers may help students explore the connections to the algebraic generalized form for finding the sum of a series of consecutive numbers $\frac{n(n+1)}{2}$ [7]. Other possible explorations relating to this are finding the sum of the first n consecutive even integers n(n + 1), finding the sum of the squares of the first n consecutive natural numbers $\frac{n(n+1)(2n+1)}{6}$, or finding the sum of the cubes of the first n consecutive natural numbers $[\frac{n(n+1)}{2}]^2$. With these ideas, it is no surprise that the sum of odd integers would be a motivating topic for students.

The Sum of the First n Odd Positive Integers is Always a Square

Mathematical induction is a technique of proof that occurs quite frequently is mathematics. Here we will assume only that if one case is true, then the next case is also true.

Let S(n) = the sum of the first n odd numbers greater than 0.
Show that S(n) = 1 + 3 + 5 + ... + (2n - 1) = n^2.

Base case (n = 1) : S(1) = 1 = 1^2. So the result holds for n = 1.

Induction Hypothesis: Assume that S(k) = k^2.
Show that S(k + 1) = $(k + 1)^2$.
S(k + 1) = 1 +3 + 5 + ...+ (2k - 1) + (2(k + 1) - 1)
= S(k) + 2(k + 1) - 1 by definition of S(n)
= k^2 + 2(k + 1) - 1 by induction hypothesis
= k^2 + 2k + 1 simplification
= (k + 1)(k + 1) factoring
= $(k + 1)^2$ simplification

Therefore, we can conclude that since S(1) = 1 and that S(k) ⇒ S(k + 1) , then S(n) is equal to the sum of the first n odd numbers for all n>0 [2].

Location of the 17 Symmetry Groups

Using the second image of Figure 2 as a pattern, it was easy to establish how to quilt 16 of the symmetry groups. I would highlight one symmetry group per cell. In this particular image, n = 4, illustrating 1 + 3 + 5 + 7. For this scheme, n=4 and the last odd integer is represented by 2n - 1, or 2(4)- 1 or 7. This would only allow me to quilt 16 of the 17 symmetry groups. To include the 17th group, it would be necessary to quilt the p6m symmetry group on the border of the quilt. If one looks closely at the border, p6m is sewn there.

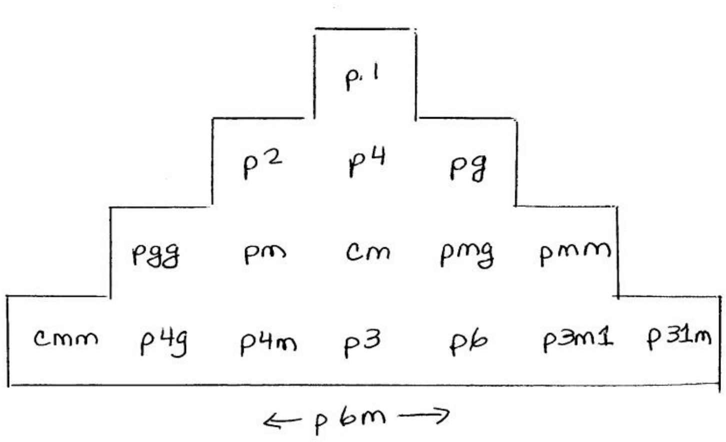

Figure 3: *The location of 17 symmetry group patterns [1]*

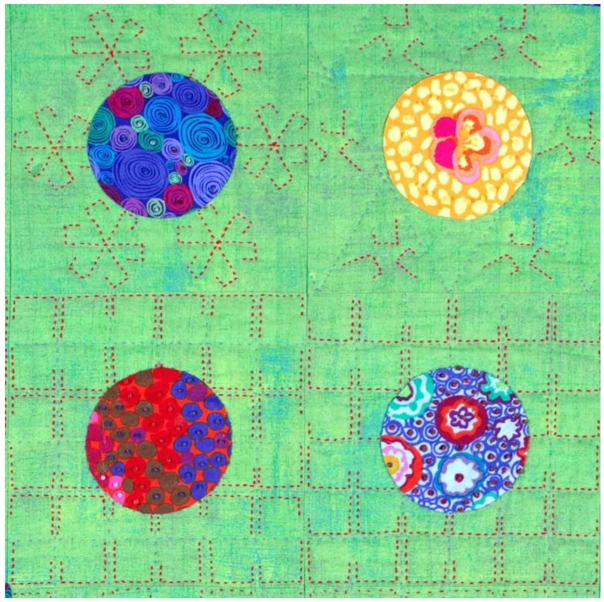

Figure 4: *Close-up of the quilting stitches in The Sum of Odd Integers--p6 ,p3 ,pmg ,cm.*

Conclusion

The topic of odd integer summation was an interesting quilt to complete and add to my collection of 65 mathematical quilts. Sometimes the pattern motivates me, sometimes the fabric is interesting, and sometimes the formula intrigues me. For this quilt, all three ideas engaged me.

Acknowledgements

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References

[1] J. Beyer. *Designing Tessellations--the Secrets of Interlocking Patterns,* NTC Contemporary Publishing Group, Inc., Chicago, 1999, pp. 100-102.

[2] Induction. www.cs.cornell.edu/courses/cs211/2005sp/Sections/S2/Induction.pdf.

[3] M. A. Lerma. https://sites.math.northwestern.edu/~mlerma/courses/cs310-05s/notes/dm-induc.

[4] L. Mastin. https://www.storyofmathematics.com/19th_gauss.html, 2010.

[5] R. B. Nelsen. *Proof Without Words, Exercises in Visual Thinking,* Mathematical Association of America, 1993, pp. 71-73.

[6] NRICH team. The Millennium Mathematics Project, “Clever Carl”, https://nrich.maths.org/2478.

[7] J. M. Wilburne. https://www/nctm.org/Publications/Teaching-Children-Mathematics/Blog/The-Story-of-Gauss. 2014.