

Rhythm Domes: Applications of Euclid's Algorithm to Architecture

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Abstract

Found in Proposition 2 of *Book VII* of Euclid's *Eléments*, the procedure now known as Euclid's Algorithm computes the greatest common divisor of two numbers. Motivated by recent studies that have applied this algorithm to the analysis of rhythmic structures in music, it is extended here into the realm of architecture. A modeling algorithm is derived from the repeated subtraction method outlined in the original book. The output is then brought into a complex geometry format by using Mobius transformations to generate mappings onto the sphere. Dome-like forms are produced that await further architectonic development and evaluation.

Introduction

Euclid's algorithm is a computational procedure used to calculate the greatest common divisor (gcd) of two numbers. The crucial insight is that if the greatest common divisor divides into the initial numbers then it also divides into their difference. This can be done in a step by step fashion involving only the elementary operation of subtraction, and thus lends itself to being automated. It is also a way to distribute the remainder of two numbers for the cases where there is no common divisor other than 1. Because of this feature it was discovered to be a surprisingly effective way to analyze musical rhythms. Toussaint [3] for example has conducted a comparative analysis of several musical rhythms. Surprising similarities were found spanning different cultural, stylistic and historical musical forms.

Levels and Groupings in Euclid's Algorithm

It is this latter usage of Euclid's algorithm that sparked my interest in possibly using Euclid's Algorithm (EA) in an architectural setting. I was intrigued with the idea of taking two unrelated numbers and being able to compare them in an orderly way, distributed as Toussaint put it as evenly as possible.

The work presented here is a work-in-progress based off of Toussaint's work on this subject. A particular musical rhythmic pattern (assumed here to be cyclic) can be seen as the number of beats within a measure, and thus represented as a binary sequence. If given this information only, for example 5 beats in a measure of 8 time units, one can ask the question how would these beats be distributed within the measure as evenly as possible? The procedure to find this out begins by associating each beat with a time unit and grouping them together, leaving a remainder behind. One repeats this procedure until there is only one group of remainders left, after which all the groups can be ungrouped and the resulting sequence is asserted to be the rhythmic sequence for that particular pair of numbers. An example of the Cuban cinquillo, using Toussaint's notation, appears as : [x . x x . x x .].

After reading this and other papers on the subject [1], I wondered how the steps of EA were connected to these evenly spaced rhythms. EA works as such: given two numbers, set the larger number to m , the smaller number to n . Subtracting n from m leaves a remainder r . For the next step, if r is less than n , swap m and n and set n equal to r . Otherwise set m equal to r and leave n as is. Repeat until $n = 1$. Borrowing from Toussaint's exposition of a similar problem considered by E. Bjorkland [3], evenly distributing a sequence of one number into a sequence of another larger number (in this case 12 into 19)

can be demonstrated as a step by step sequence of groupings as shown in Table 1. The right side of the table lists the values of the EA equation, the left side the letter index for each level or step of the procedure.

Table 1: EA array with row index, number sequence and equation values. The number of shaded areas per row correspond to the equation's n value.

ROW	SEQUENCES	m - n = r
A	1 1 1 1 1 1 1 1 1 1 1 1 0 0 0 0 0 0 0 0	19 12 7
B	[1 0] [1 0] [1 0] [1 0] [1 0] [1 0] [1 0] 1 1 1 1 1	12 7 5
C	[1 0 1] [1 0 1] [1 0 1] [1 0 1] [1 0 1] [1 0] [1 0]	7 5 2
D	[1 0 1 1 0] [1 0 1 1 0] [1 0 1] [1 0 1] [1 0 1]	5 2 3
E	[1 0 1 1 0 1 0 1] [1 0 1 1 0 1 0 1] [1 0 1]	3 2 1
A	[1 0 1 1 0 1 0 1 1 0 1] [1 0 1 1 0 1 0 1]	2 1 1

Next, an algorithm is created that uses the result of EA, the last row **A** of Table 1, to generate a subsequent row's groupings as an evenly distributed sequence. This proceeds stepwise and results in an evenly distributed binary array as shown in Table 2. The EA equation at each row is recovered and the number of groupings, indicated again as contiguously shaded cells, correspond to a row's n value.

Table 2: EA-ED diagram showing an even distribution of groupings related to steps of Euclid's Algorithm. A grouping index is found at the lower right corner of the shaded cells.

ROW	SEQUENCES	m - n = r
A	1 _{A1} 0 1 _{A2} 1 _{A3} 0 1 _{A4} 0 1 _{A5} 1 _{A6} 0 1 _{A7} 1 _{A8} 0 1 _{A9} 1 _{A10} 0 1 _{A11} 0 1 _{A12}	19 12 7
B	n n _{B1} r n n _{B2} n n _{B3} r n n _{B4} r n n _{B5} r n n _{B6} n n _{B7} r	12 7 5
C	n n n _{C1} r r n n n _{C2} n n n _{C3} n n n _{C4} r r n n n _{C5}	7 5 2
D	n n n n _{D1} r r r r n n n n _{D2} r r r	5 2 3
E	n n n n n _{E1} r r r n n n n n _{E2}	3 2 1

Architectural Applications

Noting that EA-ED generates a nested sequence of groupings whose number reduces while its membership enlarges, I thought that this may be a way to handle circular architectural forms. One could orient the larger sized groupings that are fewer in number closer to the pole and thus address practical and aesthetic problems encountered in polar or spherical coordinate systems where intervals shrink as you approach a point. Thus I began thinking of how this would look and behave as a kind of dome. Given the EA-ED output of Table 2, a grouping becomes a rectangle whose width, i.e. the number of cells it includes, increments per level according to the Fibonacci Series. The number of these rectangles corresponds to the n value of EA. In this way an EA-ED array of rectangles is produced.

To go from a polar grid to a dome, a Möbius transformation [2] followed by a stereographic projection is performed. Out of the three types of mappings, elliptical, parabolic, loxodromic, I chose the latter due to the fact that I was envisioning this as a more oriented (in both a mathematical and literal

sense) type of form. After generating a particular EA-ED rectangular array and converting it into polar coordinates, the array is adjusted so that it scales and rotates onto itself in order to produce a loxodromic mapping of the array onto the sphere. The rows correspond to the y dimension of the polar coordinate system, which circle around the poles like lines of latitude. The first EA-ED row is mapped along the equator which is set at the ground plane. The following rows cover the remaining hemisphere to the pole, with rectangles growing in size but reducing in number. This characterizes the form, pattern and structure of the dome. Figure 1 shows a schematic plan view of this.

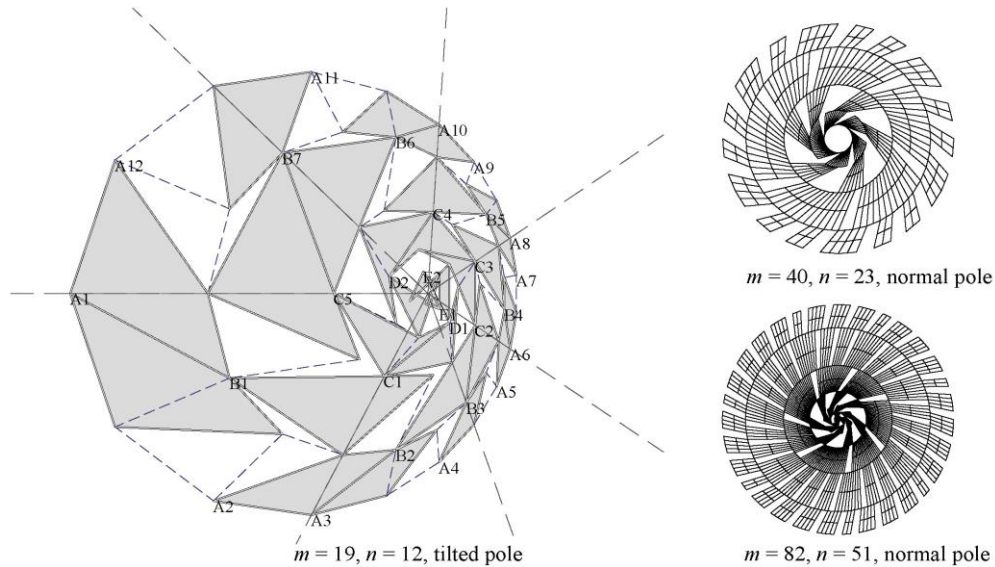


Figure 1: *Left, plan view of an array of EA-ED rectangles mapped onto the upper half of the sphere, the indices correspond to the groupings index of Table 2. Right, alternate m,n arrays.*

My first attempts in developing these schematic structures into architectural forms involved considerations of cladding and orientation. I first sent the poles of the loxodromic map onto the normal north-south poles of a sphere to generate domes with rotational symmetry and an oculus at the top. The cladding of this dome is derived from this symmetry. An example of this dome is shown in Figure 2. I then altered the diameter of the stereographic sphere so that the planar poles did not map to the poles of the sphere, resulting in domes that lose rotational symmetry, whose oculus is not at the top, and whose cladding is derived from bilateral congruency. Figure 3 depicts such a dome.

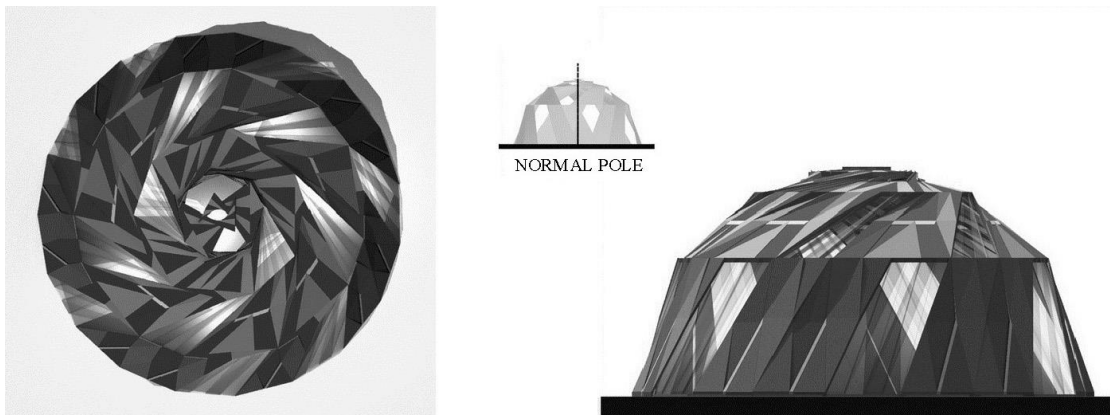


Figure 2 : *Top, plan and elevation of a normal pole dome.*

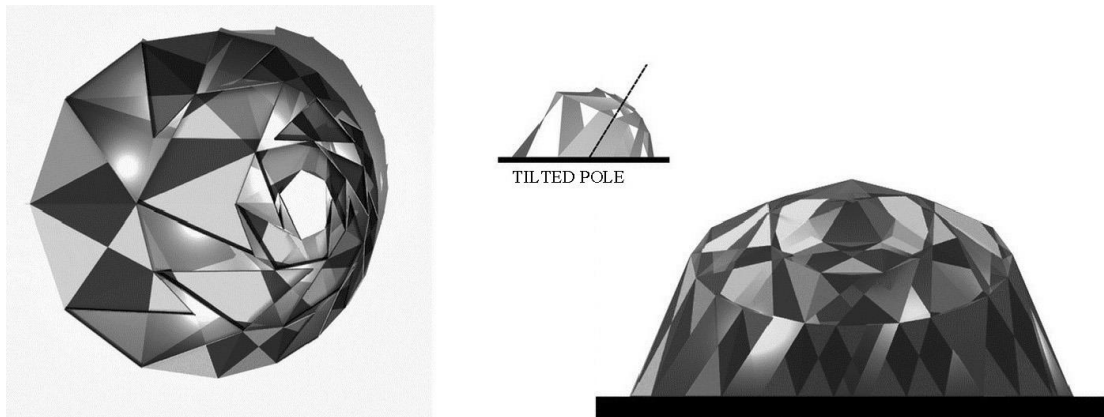


Figure 3: *Top, plan and elevation of a tilted pole dome.*

Conclusion

What began as a way to produce evenly spaced patterns was then further enriched by the elaboration of the repeated subtraction method of Euclid's Algorithm, resulting in a modeling algorithm that furnishes architectural form, pattern and structural elements as part of an integrated system. One can intervene at many levels within this system to control outcomes. Within the base algorithm, the initial choice of two numbers can be related to particulars of a given site or structure. If circular forms are desired, then one can intervene in the mapping phase and adjust the Möbius transformations that govern how points map to themselves (or not). And one can intervene in the stereographic projection phase and explore the symmetries that result. Further investigations are planned for rectilinear structures, structures that derive from the planar Möbius transformation maps, other kinds of complex mappings that generate curvilinear forms. As well the architectural ramifications of Euclid's Algorithm in the broader context of number theory [1] are only lightly touched on here, and could serve as the basis for further research.

References

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