Discrete Gyroid Surface

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Abstract

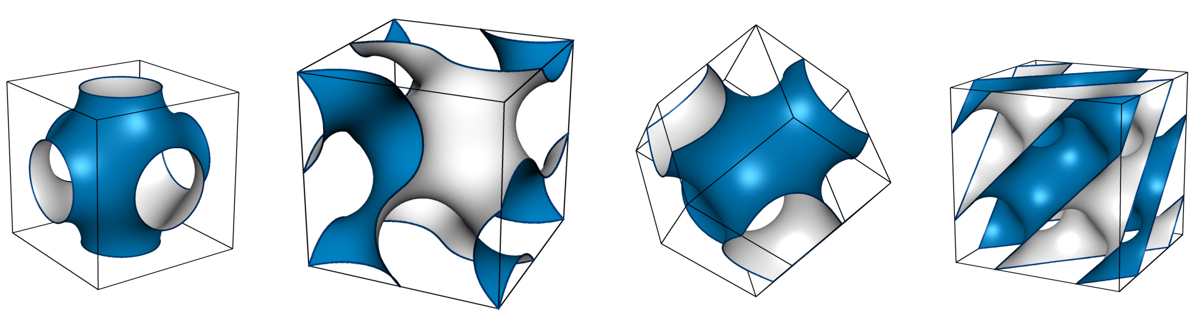
We present a discrete gyroid surface. The gyroid is a triply periodic minimal surface, our discrete version has the same symmetries as the smooth gyroid and can be constructed from simple translational units.

Minimal Surfaces, Periodicity, and the Gyroid Surface

Soap bubbles are fascinating to both children and adults. While being beautiful to look at, they solve a complex optimization problem. This becomes even more apparent when dipping a bent wire into a soap solution. The soap forms a film to span the border—given by the wire frame—with the smallest surface area possible. Famous mathematicians have worked to identify and construct different minimal surfaces, e.g. Leonhard Euler who described the Catenoid in 1744.

A large class of minimal surfaces is given by triply periodic minimal surfaces. One can think of them as constructions from translational units, where each such unit is given by a space filling polyhedron. When the surface has been constructed in one unit, it is extended to neighboring units by parallel translation. To get a continuous surface at the unit boundaries, the surface in the translational unit is often constructed with additional reflective and rotational symmetries. In the 1880s, Schwarz and his student Neovius described a set of such surfaces which results from a solution of Plateau's problem for a polygon and the aforementioned translational construction [6].

The gyroid is a minimal surface, which belongs to the associate family of the Schwarz P and Schwarz D surface, see the article of Schoen from 1970 [5]. In his publication, Schoen noted that the gyroid contains neither straight lines nor planar symmetries. He was, however, not able to prove it to be embedded in $\mathbb{R}^3$. This proof is due to Große-Brauckmann and Wohlgemuth in 1996 [1]. Like Schwarz P and Schwarz D, it is a triply periodic minimal surface. Translational units of these three infinite surfaces are shown in Figure 1.



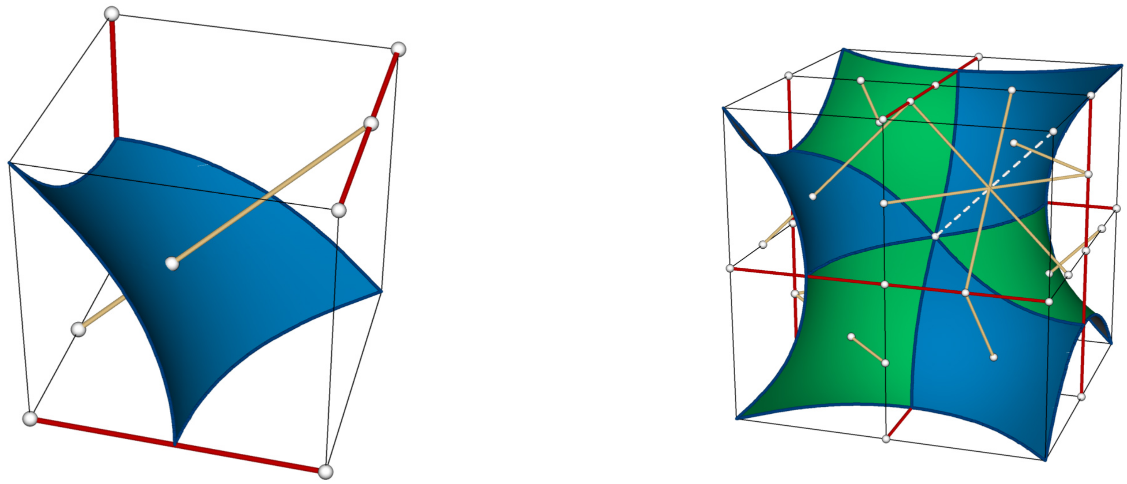
Schwarz P
cube

Gyroid
cube

Schwarz D
rhombic dodecahedron

Schwarz D
cube

Figure 1: *Translational units for triply periodic minimal surfaces*

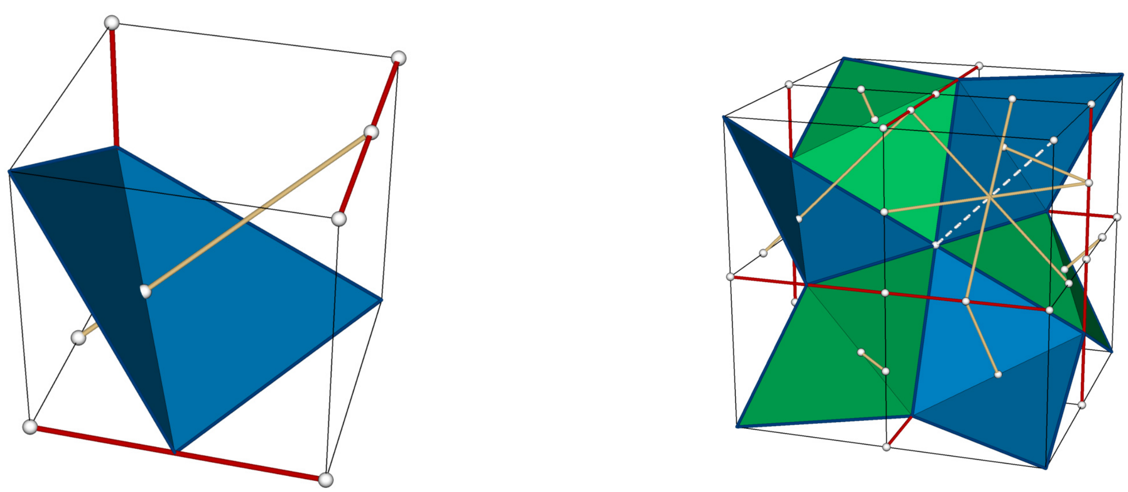


(a) *1/48 of the translational unit* **(b)** *1/8 of the translational unit*

Figure 2: *Symmetry axes of the gyroid surface, where red and yellow edges indicate axes of 2-fold rotational symmetry and the white dashed edge indicates an axis of 6-fold rotary reflection.*

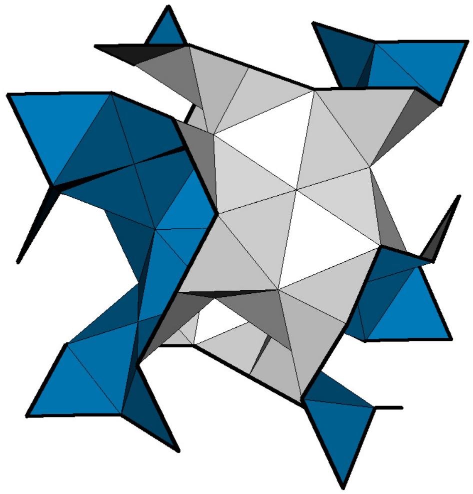
Symmetries of the Gyroid

In addition to the translational symmetry in three orthogonal axis directions, the gyroid surface has several point reflection symmetries and 2-fold and 3-fold rotational symmetries. Figure 2b shows one eights of the gyroid surface in Figure 1 with its symmetry axes: The surface has 2-fold rotational symmetry to six axes parallel to bounding box edges (colored red) and twelve axes parallel to bounding box face diagonals (colored yellow). At the intersection with the red lines, the surface has a 4-fold rotary reflection symmetry. Furthermore, the surface has point reflection symmetry and 3-fold rotational symmetry to spacial diagonals at the center and the corners of the bounding box. Figure 2a shows a smaller surface patch with four axes of 2-fold rotational symmetry. The translational unit of the gyroid shown in Figure 1 is obtained from 8 copies of the surface part shown in Figure 2b by 2-fold rotations at red edges.

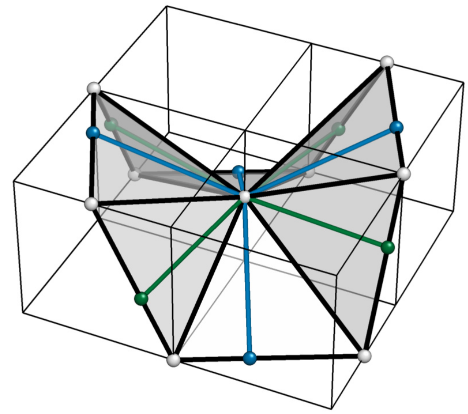


(a) *1/48 of the translational unit* **(b)** *1/8 of the translational unit*

Figure 3: *Symmetry asxes of the discrete gyroid surface, cf. Figure 2 for the smooth case.*



(a) *Translational unit of the discrete gyroid surface.*



(b) *A vertex surrounded by eight triangles.*

Figure 4: *One translational unit of the discrete gyroid surface as well as a visualization of the minimality property.*

Discrete Minimal Surface

A discrete surface is a piecewise planar surface, consisting of vertices, edges, and faces. We call such a surface discrete minimal, if the surface area cannot be decreased to first order by moving a single interior vertex, keeping all the other vertices fixed, see [4]. We can consider the total surface area as a function of all vertex coordinates. In order to obtain a discrete minimal surface, the gradient of this function has to be zero. When moving a single vertex, the area gradient is given as the sum of area gradients of the triangles adjacent to the moved vertex. These latter gradients can be computed by $\frac{1}{2}$ times the edge vector opposite to the moved vertex, rotated by $\frac{\pi}{2}$ in their triangle plane. See [4] for more details on how to compute a discrete minimal surface.

Discrete Gyroid

We will now turn to the description of the discrete gyroid surface. A surface with the corresponding symmetries can be constructed by two triangles in a box with the same size as in Figure 2a. The four vertices for the two triangles in Figure 3a are placed on two diagonally opposite corners and two edge midpoints of those edges of the box that are not incident to either of the chosen corners, cf. Figure 2 with the smooth version. Figure 3b shows a bigger patch of the surface, having exactly the same symmetries as the smooth gyroid surface, cf. Figure 2b.

A translational unit of the discrete gyroid surface is shown in Figure 4a. The surface has—up to symmetry—only two types of vertices. The first kind is surrounded by six triangles and is a center of point reflection symmetry for the surface, as indicated by the white diagonal in Figure 3b. The second kind is surrounded by eight triangles and is on a 2-fold rotational symmetry axis, consider the vertex at the center of the box as shown in Figure 4b. Each triangle has one vertex of the first kind and two vertices of the second kind, all triangles are pairwise congruent to each other.

To show discrete minimality of this surface, it is sufficient to show that the gradient of the total surface area vanishes for each of the two kinds of vertices. Since the first kind of vertex is a center of point reflection symmetry of the surface, the surface area gradient also has to be point symmetric—therefore it has to be the zero vector. For the second kind, the gradient directions for all eight triangles are shown in Figure 4b. Observe that all eight gradients have the same length since the opposite edges to the center vertex all have the same length. Due to the 4-fold rotary symmetry in the center vertex, the sum of gradients is zero in each coordinate direction.

From the translational unit of the constructed discrete gyroid, we can now easily build the whole surface. A bigger part of the surface is shown in Figure 5. Given the organic-like structure of the gyroid, it is an inspiring object that led e.g. to a sculpture by Grossman [2]. George Hart presented a different version of a discretized gyroid surface in 2010 [3]—his model does not have all symmetries of the smooth gyroid surface, but it is built from equilateral triangles only and all vertices are of the same type—a property not satisfied by our construction.

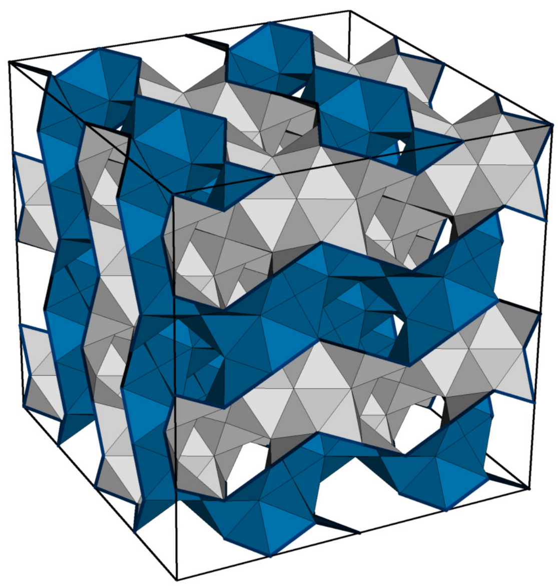


Figure 5: *Bigger part of the discrete gyroid surface.*

References

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