The Pythagorean Forest

Josef Weese[1], Joshua Martinez[2], Christian Caraveo[3], and Jesse Atkinson[4]

[1]IT, Northern New Mexico College (NNMC) josef_c_weese@nnmc.edu; [2]Engineering, NNMC, joshua_k_martinez@nnmc.edu; [3]Engineering, NNMC, christian_n_caraveo@nnmc.edu; [4]Math, NNMC, jesse_l_atkinson@nnmc.edu

Abstract

We show how a network of ideas—a forest of trees—that constitute a 3-D tree graph of a proof in geometry both interact and accumulate in number and kind. Our bar graphs, adjacency matrices, and a new 3-D glass and wood sculpture represent ways to visualize a geometry proof (in our case, the Pythagorean Theorem) and its interacting premises.

Perspective from the Tree

Proving Euclid's propositions from his *Elements* is one thing. Graphing them, on the other hand, seems a "no-go." Recently, in Math*Overflow*, a Q. & A. math website, this query appeared: "Is anyone aware of any attempt to describe the dependencies of theorems [propositions]…in the form of a family tree [tree graph]… where each node on the tree might correspond to a theorem and branches would indicate dependencies between theorems" [9]?

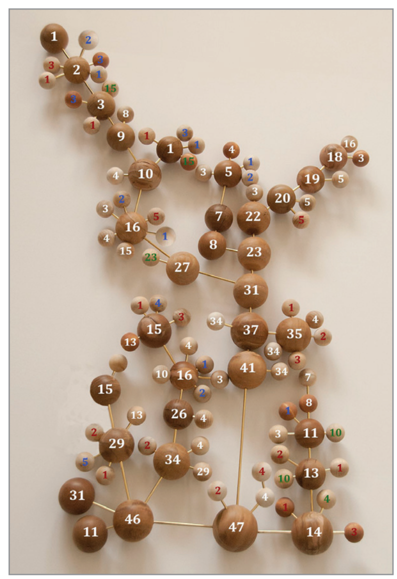

Figure 1: *3-D Tree Dependency Graph for the proof of the Pythagorean Theorem, 1.47, Euclid, Book 1. Proposition 47 is at bottom, center. Key: White – Propositions; Red – Common Notions; Blue – Postulates; Green – Definitions [2].*

The answer is: Yes! That is what we do. Our colleague, Jesse Atkinson, is the first, publishing in the Bridges Math/Art Conference Proceedings (2016) [1] and exhibiting in the Conference's art gallery his rooted and directed in-tree 3-D dependency graph for Euclid's proof of the Pythagorean Theorem, Book 1, 1.47—see numbered graph in figure 1 [2]. An in-tree directed dependency occurs when "A depends upon B if B is necessary in the proof of A" [1].

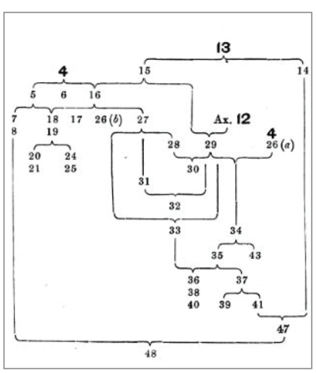

Figure 2: *From Lewis Carroll's Book, 1879, showing dependencies among all Propositions in Book 1, Euclid's Elements [5].*

Dependency graphs do exist in the literature, but they are for the whole of Euclid's Book 1—see e.g., figure 2 (Lewis Carroll) [5]. More recently, Schiefsky (2007)[11], Boxer and Clutter (2016) [4], and Boman's group (2014) [3] produce similar graphs (none shown here). However, these dependency :graphs are not for a single proof such as Euclid's 1.47, hence none is a "family tree." Put another way, there is no root in their graphs.

All edges (connections) in our 3-D graphs direct towards the root of the tree because that root is dependent upon all the premises (vertices, nodes) for its proof. These graphs are unique because there is a proof for every proposition needed to prove any other proposition, should it appear as a premise. However, if a proposition appears more than once, we cite it but do not prove it again.

Teasing the Tree

The tree graph in figure 1 displays all of the direct and indirect premises used to prove proposition, 1.47 from Heath's Euclid. These premises include other propositions from Book 1, one proposition from Book 2 in some models, as well as some or all of the Common Notions, Postulates, and Definitions from Book 1.

Our project is to "tease" apart Euclid's proofs and look at each premise's appearance. For example, we ask how many times (number of appearances) Proposition 1.1 appears in the expanded proof and give it an A-K number—see figure 3. Also, we determine at what level a premise makes its first appearance and what its highest level is, e.g., its furthest distance from its root? There are seven levels in the proof for 1.47.

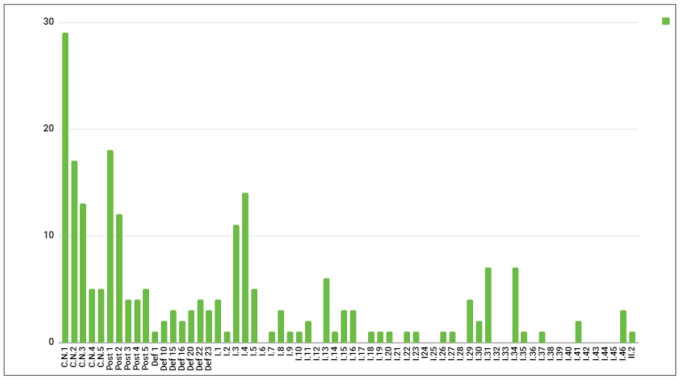

Figure 3: *A-K Collaboration graph showing number of appearances in Euclid's proof for all direct and indirect premises for 1.47, Book 1, Elements. A-K Numbers, in analogy with Erdös numbers, indicate the number of citations or collaborations in a proof for a given premise. "A" is Atkinson; "K" is Knight. The A-K Number is a new measure we introduce into the literature.*

By "level," we understand the number of nodes or vertices that the dependency passes through from the root to a given premise, i.e., the passage from the root through the direct and the indirect premises. We display the data for the number of appearances in the bar graph in figure 3.

We compile our data from our generic 2-D tree graph (not shown) that includes the suggested annotations for premises for the Pythagorean Theorem (1.47) from four sources—Heath [8], Heiberg/Fitzpatrick [7], Clark University [6], and Neuenschwander [10]—on the principle that "more heads are better than two." Not surprisingly, not all agree on what constitutes a premise in Euclid's proof. Playing no favorites, we use all of their suggestions except those of Neuenschwander for the Common Notions—he substitutes for these what he calls "axioms" and re-writes that part of Euclid's text with some "new" axioms. We keep Euclid's original five Common Notions in our data and count Neuenschwander's citations only when his axioms correspond to one of Euclid's five Common Notions. Taken as a whole, these annotations constitute 1.47's toolbox—its repertoire, as we like to call it. It is our generic data.

In addition to constructing bar graphs, we present our data in adjacency matrices (fig. 4), another format for displaying the relationships among the premises of proven propositions. Schiefsky [11] does a similar matrix for all of Book 1 with axes being propositions, but nowhere have we found a matrix similar to figure 4 where one axis is propositions and the other is Common Notions, Postulates, and Definitions. In fact, other dependency graphs do not include any of these latter three types of premises, though for completion of the proof tree, we believe they should.

	P1	P2	P3	P4	P5	P6	P7	P8	P9	P10	P11	P12	P13	P14	P15	P16	P17	P18	P19	P20	P21	P22	P23	P24	P25	P26	P27	P28	P29	P30	P31	P32	P33	P34	P35	P36	P37	P38-40	P41	P42-45	P46	P47	P48	PII-2
CN1	1	1	1				1						1	1	1					1		2				2			2	1					1		1		1		1	1		1
CN2													2	1	1					1									1					1	2				1			2		1
CN3		1			2									1	1																				1						1			
CN4				4				1																																				
CN5							2									1				1																						1		
Post1	1	1		1	1		1	1	1		1					2		1		1		1	1			2					1			1					1			1		
Post2		1		1	1			1						1		1				1							1		1		1						1							
Post3	1	3	1																			1																						
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Post5																													1								1				1	1		1
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Figure 4: *Modified Adjacency Matrix showing distribution of Common Notions, Postulates, and Definitions throughout the propositions used in Euclid's proof of the Pythagorean Theorem, 1.47. If a premise appears more than once in a proof, we cite it. In some cases, there are four citations. Abscissa=Propositions, Ordinate=C.N., P., and D.*

Schiefsky's matrix for Book 1 [11], mentioned above, shows results very similar to ours for a Propositions X Propositions matrix (neither shown here). However, our data is for the Pythagorean Theorem alone. Given the amount of space 1.47 takes in Book 1, perhaps this is not unexpected.

A Forest among the Tree

Being mindful that truth is beauty and that our original tree graph in figure 1 contains several kinds of premises, we move beyond matrices and bar graphs and use our 2-D generic data in a new 3-D sculpture of Euclid's proof of Pythagorean Theorem—figure 5. Our original tree becomes a forest of trees.

Each kind of premise appears separated from the others in the seven levels that the proof occupies. For example, Definitions in levels 2, 4, 5 and 7, appear as the smallest wooden spheres. Similarly, if Technological Q.E.F. (Construction) Propositions (e.g., Euclid's 1.1) are in levels 3, 5, and 7, they appear there as larger wooden spheres. These new "trees" are not dependency tree graphs in the sense understood up to now. They are just trees, made up of similar kinds of premises that occur at different levels.

Three other kinds of "trees" exist in the forest—Logical Q.E.D. (Inference) Propositions, Common Notions, and Postulates. In total, there are 26 construction propositions, 63 logical propositions, 15 definitions, 43 common notions, and 45 postulates in the 5 trees in our forest.

All these premises and their root occur in figure 5, our prototype sculpture. Each of the seven levels is a glass surface, 24" x 18", with an overall height of 16.5", pillared by 2" wood cubes. At ground level, there are 5 wooden spheres (the largest ones), each representing the root of their tree, 1.47. All spheres and pillars are either maple (Acer) or birch (Betula), two genera common in ancient Greece. Acrylic plates replace the glass surfaces when the sculpture travels, and another version color codes the 5 kinds of trees.

An unexpected consequence of the new 3-D sculpture is that the spheres at each level reflect below that level's surface, as if reaching towards its predecessor, as one might expect given the in-root form of the dependency. Viewed from the top, the spheres appear as a virtual universe floating in space. As in any forest, the levels interact with each other, from the ground to the canopy.

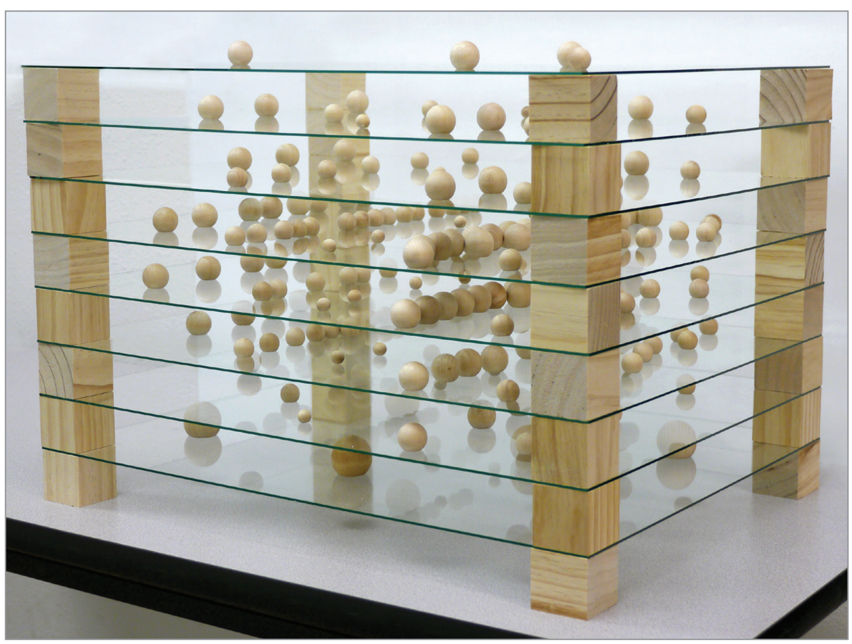

Figure 5: *3-D Prototype Forest Graph sculpture in wood and glass of the network of premises in Euclid's proof of the Pythagorean Theorem, Book 1, 1.47. See details in previous section.*

View from the Canopy

We are aware that there are many proofs for the Pythagorean Theorem; even Euclid has another in Book VI (6.31). However, we claim that if Book 1's proof of 1.47 is invalid, so are all its other proofs. Be that as it may, we ask: What does our Pythagorean Forest teach us about this proof and, by implication, others?

A sole consideration of the direct premises of a proof such as Euclid's 1.47, for which there are 14 by our count, obscures the role played by the much larger number of indirect premises, of which there are approximately 178, depending upon who is counting. We teased apart these latter, noting their different kinds and analyzing their relationships to the root proposition that depends on them through our two 3-D tree sculptures, bar graphs, and adjacency matrices.

Premises are ideas. We ask the significance that Common Notion (C.N.) 1 appears more often than any other C.N., 29 times, in fact. Postulate (P.) 1 appears 18 times, more than 3 times as often as P. 4, and 6 more times than P. 2. There is a hierarchy here. Can one claim that a premise that appears least often is less important than one that appears more often? If so, important in what way—a more fundamental idea? We calculated the mean, mode and median distances for all premises but do not present that data here; again, this is data needing interpretation. We plan to do similar analyses for the structure of Euclid's proof, 1.45, and already have a prototype 3-D directed dependency graph sculpture for that proposition.

Acknowledgements

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