Braids Formed by the Impression of Knots

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Abstract

The inhabitants of ancient Mesopotamia used small objects known as cylinder seals with carved figures or motifs to make impressions onto clay. Among these motifs and patterns there are specimens showing knots and links carved on the surface of the seals. However, when unrolled, some of these patterns form braids. Here we give examples and visualise the connection between knot theory and braid theory, using archaeological means.

Introduction

Around 3500 - 3200 BC in Mesopotamia the cylinder seal first appeared and succeeded other means of identification like stamp seals. By rolling the cylinder seal on wet clay, a laterally reversed impression of the inscription or markings are impressed into the clay, which will dry/burn and thus claim the ownership of the artefact. These cylinder seals are often rather small, about 15 - 40 mm in length and 8-25 mm in diameter. They are made of hematite, steatite, lapis lazuli, limestone, etc. and encountered by archaeologists in plenty. They are nowadays believed to have been used basically by everyone in Mesopotamia. In order to use such items for identification and safeguarding important goods from tampering they need to be more or less unique. Among the many samples of cylinder seals, there are specimen decorated with figurative motifs, but there are also examples decorated with geometrical patterns. Such examples of cylinder seals with carved geometrical patterns can be found which originate from the early dynastic period, 2900 - 2334 BC. Among these geometrical patterns, there are examples of projections from knots and links that when unrolled, form braids, like the example in Figure 1. These projections, sometimes rather complex, give witness to the skills of our earlier ancestors. In order to manufacture these patterns it is likely that some kind of knowledge in geometry or at least a mathematical thinking was needed. In the 3rd millennium BC the mathematical study of geometry was in its infancy, as evidence carved on Egyptian and Babylonian clay tablets show us. It is thus even more intriguing to think about the sense for mathematical thinking these craftsmen must have had. In the subsequent section we look briefly at the mathematical tools at hand as of today for studying such patterns. The interested reader may want to read [8] in order to experiment and create their own seals.

(a) (b)

Figure 1: *Modern made cylinder seal and corresponding impression: (a) cylinder seal made by C. Åström featuring a knot and (b) impression on clay featuring a braid.*

Topology

Topology is the mathematical study of objects, independent of their size, which can be exposed to operations such as twisting or bending and still being topologically equivalent. Thus being preserved under continuous deformations, see [2] among others. Topology, as a mathematical discipline, goes back to 18th century and works by for instance Euler [6]. In this paper there are two branches of topology of interest to us, knot theory and braid theory. These are briefly introduced in the following two subsections. The outset of knot theory was made by the works of for example Gauss [7] and braid theory by Artin [3].

Knot theory

A knot is like a piece of string where the ends are joined, thus a knot K is a closed loop curve in $\mathbb{E}^3$. One or more *interlinked* knots is called a *link* L, which is a set of disjoint knot(s) in $\mathbb{E}^3$. The number of knots in a link are referred to as *components* and the number of components is the *multiplicity* $\mu(L)$ of the link. For further details see [1] among others. An important type of knot for this paper is the Turk's head knot $H_{p,q}$, which is a periodic knot that can be characterized by its number of bights p and leads q. The bights are the segments of rope that form the boundary of the knot and give its scalloped appearance. The leads are the individual strands along a ray emanating from the centre of the knot. A Turk's head knot can be represented both in flat form, as in Figure 2 and Figure 5a, and in the form of a cylindrical shell as in Figure 1a. A Turk's head knot $H_{p,q}$ will be made from GCD(p, q) distinct components, i.e. the multiplicity. See for instance [9] and [11] regarding the naming of these knots and in general [4] and [10].

Figure 2: *The trefoil knot* 3_1*, also known as a Turk's head knot* $H_{3,2}$.

Braid theory

A braid can be described by a set of points $A = \{a_1, a_2, \ldots, a_n\}$ on the side of a rectangle (which frames the braid) and set of points $B = \{b_1, b_2, \ldots, b_n\}$ on the opposite side of the rectangle. Each a_i with $i \in A$ is connected by a string to a unique b_j with $j \in B$. The direction of each string, in relation to the two opposite sides of a rectangle in which the sets A and B belongs to, always heads the same way. For a further description see [3] among others. A braid with n strings can be identified using a *braid word*, which is an ordered combination of operators σ_i $(i = 1, \ldots, n-1)$ and σ_i^{-1} $(i = 1, \ldots, n-1)$. The operators indicates, respectively, if two strings i and $i+1$ passes over or under each-other. Braids as opposed to knots, consists of open strings, however, the closure of a braid constructs a link. This gives the least number of tangles to obtain the closed braid, which is known as the *braid index*. Figure 3 shows an example of a braid connecting strings according to; $\{a_1, b_1\}$, $\{a_2, b_3\}$ and $\{a_3, b_2\}$. This braid can be identified by the braid word $\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_1^{-1}$.

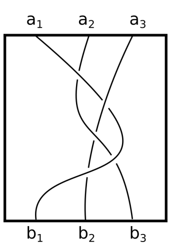


Figure 3: *Example of a braid,* $\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_1^{-1}$.

Archaeological aspects

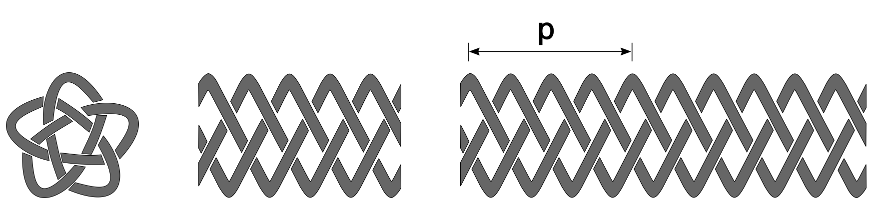
The introduction of Turk's heads, in the form of a cylindrical shell, carved on the surface of cylinder seals some 4500 years ago in Mesopotamia, marks an event in history, which seemingly have continuously been practised ever since. The imprints of Turk's head have ever since been made on a large amount of materials and different type of artefacts. In [5] we give an overview of Turk's heads, among other knots, from an archaeological and historical point of view. Despite of being mathematically equivalent, there are naturally some variations in these patterns. In the literature, however, they are treated and named rather differently. Using mathematics, specifically knot theory, we can classify these knots and links on the artefacts and more easily compare them. For example studying their evolution and hence growing complexity. In addition, consider the early specimen of cylinder seals that have Turk's heads carved on the surface, such as the $H_{3,4}$ shown in Figure 4a, and the potential artefacts they give rise to. That is, the actual piece of clay which holds an impression of a cylinder seal being artefacts on their own. In fact, clay impressions like these, showing braids are also archaeologically encountered. A modern impression of a cylinder seal featuring a knot is seen in Figure 4b, presenting a braid, impressed on clay. Archaeologically encountered cylinder seals show examples of Turk's heads in the form of a cylindrical shell with various multiplicity (e.g. 1, 2, 3). Further, clay artefacts like this do not necessarily come with an actual corresponding cylinder seal. Some may be matched to an already known cylinder seal and some do not. Not knowing the exact size and when the pattern starts to repeat (i.e. its periodicity), which may be the fact if the impression is fragmented for instance. It may then be a good idea to classify the pattern as a braid as we do not know the actual knot or link. In some cases other information in the impression than the braid itself may give us a clue regarding the cylinder seal and the corresponding knot or link. We may then use the braid closure to construct a link and thus pair the braid and the link.

(a) (b) (c)

Figure 4: *Near eastern cylinder seal with corresponding impression on clay: (a) cylinder seal featuring a $H_{3,4}$ knot, (b) impression on clay featuring a braid and (c) the corresponding braid diagram.*

Discussion

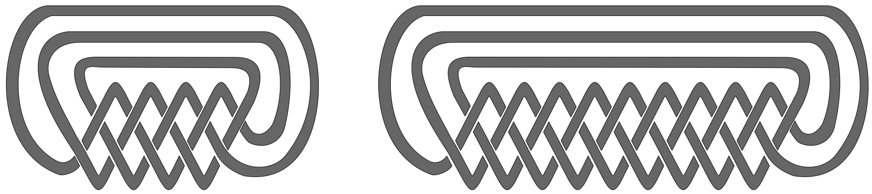
We have briefly looked at knot theory, braid theory and some archaeological aspects where both disciplines to some extent is applicable. In Figure 5a a flat Turk's head knot $H_{5,3}$ is shown, this knot can be represented as a cylindrical form like the one seen in Figure 1a. Consider the cylindrical representation of $H_{5,3}$ carved on a cylinder seal, it will form the braid $\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}$ when unrolled one full revolution, see Figure 5b. It a more simplified form it could be written as the braid word $(\sigma_1\sigma_2^{-1})^5$. One full revolution of the knot can be seen as the period length p of the braid. Figure 5c shows a braid where the same cylinder seal has been unrolled two full revolutions. The braid word then repeats $(\sigma_1\sigma_2^{-1})^5(\sigma_1\sigma_2^{-1})^5$ or $(\sigma_1\sigma_2^{-1})^{10}$, confirm by looking at the period length. The corresponding Turk's head knot that would have created the $(\sigma_1\sigma_2^{-1})^{10}$ braid with just one revolution of the cylinder seal is instead the knot $H_{10,3}$.



(a) (b) (c)

Figure 5: *Examples of knots and braids: (a) flat Turk's head knot $H_{5,3}$, (b) braid corresponding to the cylindrical representation of $H_{5,3}$: $(\sigma_1\sigma_2^{-1})^5$ and (c) $(\sigma_1\sigma_2^{-1})^{10}$.*

Applying the braid closure on the example in Figure 5b, the $(\sigma_1\sigma_2^{-1})^5$ braid, will make a closed braid representation of the knot $H_{5,3}$, see Figure 6a. The $(\sigma_1\sigma_2^{-1})^{10}$ braid, will make a closed braid representation of the knot $H_{10,3}$, see Figure 6b. Note that both braids have the same braid index. The braid index invariant can be useful if for instance having only the braid impression and not the cylinder seal with the knot or link.



(a) (b)

Figure 6: *Closed braid representations of knots: (a) $H_{5,3}$ and (b) $H_{10,3}$.*

Conclusion

In this paper we have looked at the ancient tradition and practice of unrolling cylinder seals onto for example clay tablets or the equivalent, with the result of laterally reversed impressions of the pattern on the seal. The specific pattern of interest, Turk's heads knots and links, when unrolled, forms braids. By modern mathematical tools we can see an archaeological connection between the mathematical branches of topology; knot theory and braid theory. More specifically, how these ancient cylinder seals form a 'natural' bridge between the two mathematical instances, and this, several thousands of years ago. In our opinion, this gives a very interesting and noteworthy addition to the history of topology.

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