New Generalizations of Quadratic Julia Sets to 3D

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Abstract

We present a novel construction of three-dimensional generalizations of complex quadratic Julia set fractals. While some extensions exist in the literature, using, e.g., quaternions, none have been able to extend the intricate fractal nature to higher dimensions. Here we present a new approach which is based on the so-called Sullivan's dictionary, which builds analogies between the fields of complex dynamics and Kleinian groups. Here Julia sets correspond to Kleinian limit sets, which have known extensions to 3D. By taking a special Kleinian group and its 3D extension, we can obtain information about the sought generalization of Julia sets. This leads to two extensions – the simpler 'inflated' Julia sets, and truly fractal 3D Julia sets which we were able to construct in several special cases.

Mandelbrot and Julia sets

The Mandelbrot set is one of the most recognizable, visually intriguing complex objects in mathematics. As such, it has played an important role in the popularization of mathematics as well as raising interest and attracting prospective students. The Mandelbrot set is defined by an elementary iterative procedure in the complex plane which, although very simple, gives rise to an extremely complicated intricate fractal object with deep mathematical properties, cf. [4]. The Mandelbrot set itself represents a 'catalogue' of quadratic Julia sets. The basis of these sets is the study of the iteration of a simple quadratic mapping

$$f_c(z) = z^2 + c \tag{1}$$

in the complex plane $\mathbb{C}$, where $c \in \mathbb{C}$ is a given constant. The filled-in Julia set of f_c is the set of all $z_0 \in \mathbb{C}$ such that the iterates $z_{n+1} = f_c(z_n)$ do not converge to infinity. The *Julia set* is then the boundary of the filled-in Julia set, cf. Figure 1 (bottom) for $c = -1$, the so-called *Basilica* set. Finally, the Mandelbrot set is the set of $c \in \mathbb{C}$ such that the Julia set of f_c is a connected set.

Since its discovery, people have tried to generalize the Mandelbrot set to higher dimensions than the 2D complex plane. Disappointingly, the natural extension of (1) from $\mathbb{C}$ to the four-dimensional algebra of quaternions is trivial, the resulting Mandelbrot and Julia sets are simple 4D rotations of the complex sets. Many other possible generalizations of (1) have been tested, based on exotic algebraic structures (bicomplex numbers, Clifford algebras, ...), geometric heuristics mimicking the behavior of (1) (Mandelbulb, Mandelbox, ...), etc., cf. [2]. None of these generalizations were able to extend the intricate fractal nature of the complex case as well as produce mathematically interesting objects. Here we present possible generalizations based on well established connections between the fields of complex dynamics and Kleinian groups.

Kleinian limit sets

We consider *Möbius transformations*, i.e., functions of the form $\frac{az+b}{cz+d}$ acting on $\mathbb{C}$, where $ad - bc \neq 0$ to ensure the mapping is bijective. These mappings can alternatively be characterised as a composition of a finite number of circle inversions (here a line is considered as a circle with infinite diameter). Without going into technical details, a *Kleinian group* Γ is a discrete group of Möbius transformations. In our context, we consider groups consisting of a finite number of Möbius transforms (generators) along with all possible compositions of these functions and their inverses.

If one takes an arbitrary point $w \in \mathbb{C}$ and applies each $g \in \Gamma$ to it, the resulting points will accumulate on the *limit set* of Γ. Similarly to Julia sets, Kleinian limit sets are very beautiful objects with a complicated fractal structure. Figure 1 (top) is an example of a simple limit set of a group with two generating functions.

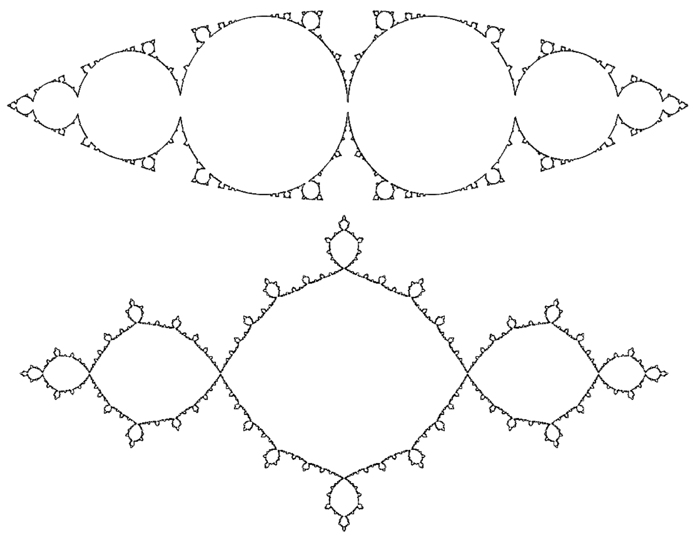

Figure 1: *Kleinian limit set (top) and Basilica Julia set (bottom).*

From Figure 1 it seems that there must be some underlying deeper connection between Julia sets and Kleinian limit sets, as they possess similar structure. Indeed this is the case as realised by Dennis Sullivan in the 1980s – the so-called Sullivan's dictionary translates corresponding concepts and theorems from the two respective fields. Here Julia sets of rational maps correspond to Kleinian limit sets, cf. [4] for an overview.

The idea presented in this contribution is that while natural higher-dimensional extensions of Julia sets are unknown, for Kleinian limit sets such generalizations are well known, cf. [5]. Figure 4 (top) shows a possible extension of the limit set from Figure 1 to 3D. We encourage the reader to examine the wonderful pictures of these sets created by Jos Leys, [3]. Hopefully, one could use the known 3D generalizations of Kleinian limit sets to gain insight or even construct the unknown 3D Julia sets.

Construction of generalized 3D Julia sets

The key to the chaotic and fractal nature of quadratic Julia sets is the function z^2 which figures in (1) and most attempts to generalize Mandelbrot and Julia sets focus on finding a suitable generalization of the complex function z^2 to the three-dimensional space. In our case, we wish to gain insight about such a generalization from the connection with Kleinian limit sets. We note that in the first step, one can consider the action of z^2 on the unit circle in $\mathbb{C}$ – here z^2 corresponds to doubling the angle of z when taken in polar form.

One must first endeavor to explicitly construct this connection in the complex case, where both Julia and Kleinian limit sets are known and then extend the Kleinian group to 3D. To this end, we consider a special Kleinian group Γ with two generating functions:

1. g_1 is the inversion in circle C_1 with center -2 and radius $\sqrt{3}$ composed with reflection in the real line,

2. g_2 maps the interior of the circle C_2 with center $1+\sqrt{3}i$ and radius $\sqrt{3}$ to the exterior of the circle C_3 with center $1-\sqrt{3}i$ and radius $\sqrt{3}$ and vice versa. The configuration is sketched in Figure 2 (left).

Both g_1, g_2 and their inverses preserve the unit circle C_0 in $\mathbb{C}$, which is also the limit set. If we denote the intersection points of C_1, C_2 and C_3 with C_0 as $A = 1, B = -\frac{1}{2}+\frac{\sqrt{3}}{2}i$ and $C = -\frac{1}{2}-\frac{\sqrt{3}}{2}i$, then $g_2(A) = A = A^2$, $g_1(B) = g_2(B) = C = B^2$ and $g_1(C) = g_2(C) = B = C^2$. Therefore, on A, B, C, the functions g_1 and g_2

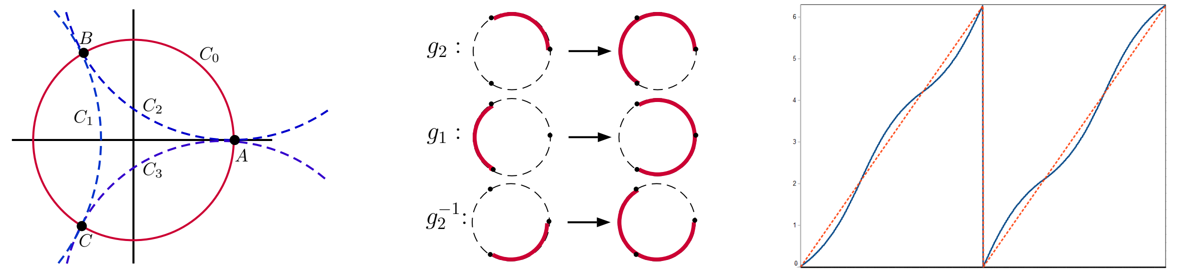

Figure 2: *Left: Construction of generators. Center: Action of generators on individual arcs. Right: Squaring map on the unit circle (red) and its Kleinian approximation (blue) as maps on* $[0, 2\pi)$.

coincide with the function z^2. On the rest of C_0, g_2 maps the shorter arc AB to the longer arc AC in a one-to-one fashion. Similarly, g_1 maps the shorter arc BC to the longer arc CB and g_2^{-1} maps the shorter arc CA to the longer arc BA, cf. Figure 2 (center). One can easily see that the function z^2 maps the mentioned arcs in the same way. Therefore, using g_1, g_2 and g_2^{-1}, one can construct a function h on C_0 which behaves like z^2, cf. Figure 2 (right). Specifically, it can be proven that $f(z) = z^2$ and h are topologically conjugate, i.e., there exists a homeomorphism φ on C_0 such that $h = \varphi^{-1}(f(\varphi))$ on C_0. This implies by induction that the iterates of h and f are in the same relation: $h^{\circ n} = \varphi^{-1}(f^{\circ n}(\varphi))$, hence the two functions have the same behavior under iteration, cf. [1]. The behavior of z^2 outside the unit circle can be obtained by simple radial scaling.

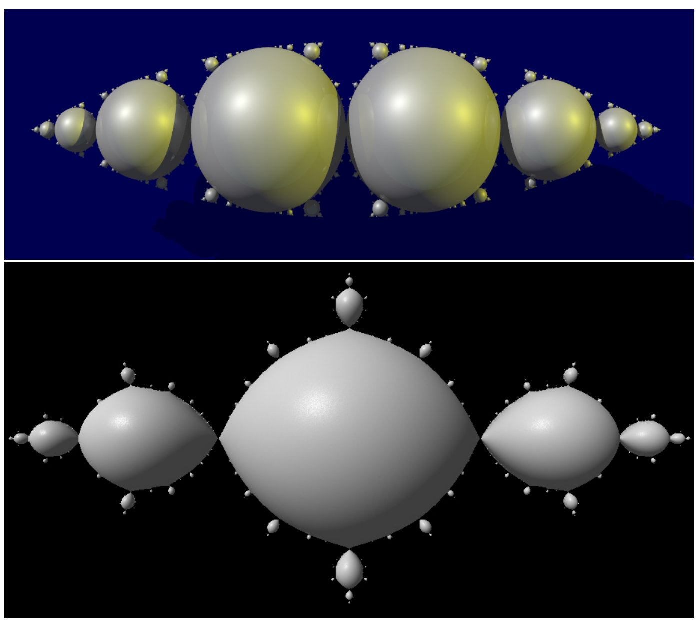

Figure 3: *3D invariant set of Kleinian group (top) and "inflated" Julia set (bottom).*

Now the group Γ can be simply extended to 3D, where inversions in circles are replaced by inversions in spheres with the same center and radius. If we denote the sphere with center X and radius r by $S_r(X)$, then we consider two possible extensions of Γ to 3D:

1. $\tilde{\Gamma}_1$: g_1 is the inversion in $S_{\sqrt{3}}(-2,0,0)$ and reflection in $\{y=0\}$, g_2 maps the interior of $S_{\sqrt{3}}(1,\sqrt{3},0)$ to the exterior of $S_{\sqrt{3}}(1,-\sqrt{3},0)$ and vice versa.

2. $\tilde{\Gamma}_2$: to $\tilde{\Gamma}_1$ we add a third generator g_3 which maps the interior of $S_{\sqrt{3}}(1,0,\sqrt{3})$ to the exterior of $S_{\sqrt{3}}(1,0,-\sqrt{3})$ and vice versa.

Both $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ preserve the unit sphere, which is the limit set of $\tilde{\Gamma}_2$, but only an invariant set of $\tilde{\Gamma}_1$ (the limit set is *smallest* invariant set, i.e., the unit circle). Going from Γ to $\tilde{\Gamma}_1$ and finally $\tilde{\Gamma}_2$ can be demonstrated on the example of the limit set from Figure 1, which unlike the unit circle and sphere has a fractal nature. Then going from Γ to $\tilde{\Gamma}_1$ corresponds to the invariant set in Figure 3 (top), where the fractal nature is essentially two-dimensional. Finally, going from Γ to $\tilde{\Gamma}_2$ gives the limit set in Figure 4 (top), where the fractal nature is truly three-dimensional.

One can now endeavor to gain insight on the dynamics of possible generalizations of z^2, namely on the unit sphere, similarly as above. This leads to two possible 3D versions. First, the so-called 'inflated' Julia sets, which stem from considering the dynamics of $\tilde{\Gamma}_1$ on the unit sphere. These mimic the essentially two-dimensional fractal nature of the resulting 3D object, Figure 3. Second, by considering the dynamics of $\tilde{\Gamma}_2$ on the unit sphere, one can obtain truly 3D fractal Julia sets as in Figure 4, where a 3D generalization of the Basilica set is presented. While the 'inflated' Julia sets can be constructed for $z^2 + c$ for arbitrary $c \in \mathbb{R}^3$, so far the second construction is unfortunately limited to $c \in \mathbb{R}$. Due to the rather technical nature of these considerations and the resulting constructions, we omit them from this short note. The work is in progress.

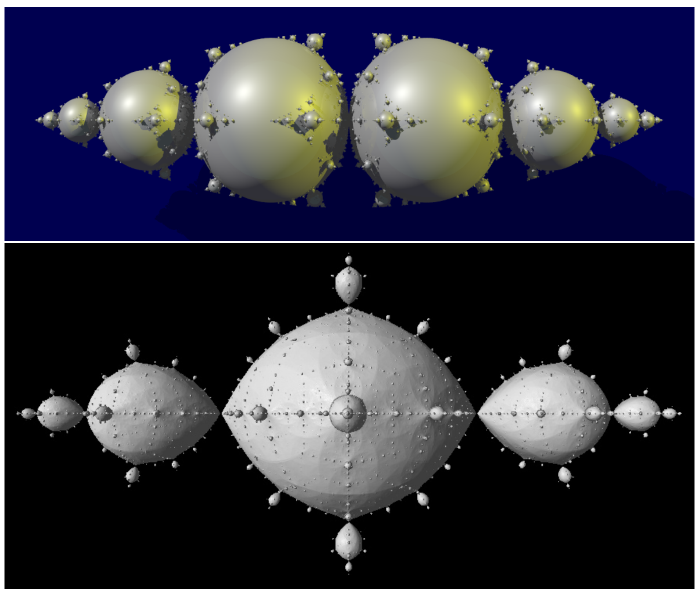

Figure 4: *3D Kleinian limit set (top) and 3D Julia set (bottom).*

Acknowledgements

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