Curved Crease Folds of Spherical Polyhedra with Regular Faces

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Abstract

Based on the art of curved crease origami, we give an example of the construction of a family of shapes from mathematically ideal paper that consists of cylinders and cones and reassembles the structure of a spherical polyhedron with regular faces. We offer explicit formulas to parametrize the crease curves. Moreover, we illustrate this method on the five Platonic solids, which can be folded from one single sheet of paper.

Introduction

This paper is a continuation of our work on mathematical curved crease paper folding [6, 7]. Motivated by the properties of real paper, we model an ideal paper as an infinitesimally thin shape that can be obtained from a planar patch without stretching or tearing, i.e., as a composition of developable surfaces. The fundamentals for mathematical paper folding are given by Huffman [5] and Fuchs and Tabachnikov [4]. Further properties were investigated by Demaine et al. [2, 3].

However, given a folded shape and its development, it is most of the times unknown whether the given shape would exist in the mathematical world as real paper seems to allow little imperfections. For example, Demaine et al. [1] show that the so-called pleated hyperbolic paraboloid does not exist without additional creases. A positive result on the other hand is the folded Vesica Piscis, cf. [6].

In this paper we give a construction method for a family of shapes that consist of planar, cylindrical and conical patches and are based on spherical polyhedra with regular faces, such as the Platonic and Archimedean solids. Experimental studies of folded Platonic solids were pursued by Schling and Otterson [8]. However, their approach is just an approximation of a sphere. Our method on the other hand starts with right circular cylinders whose profile curves are the resp. great circles on the sphere, i.e., the geodesics connecting two points of a spherical polyhedron. Those cylinders are then folded into cones with appropriately chosen apices. Moreover, we then can add additional planar creases that reflect the interior cones, see Figure 1.

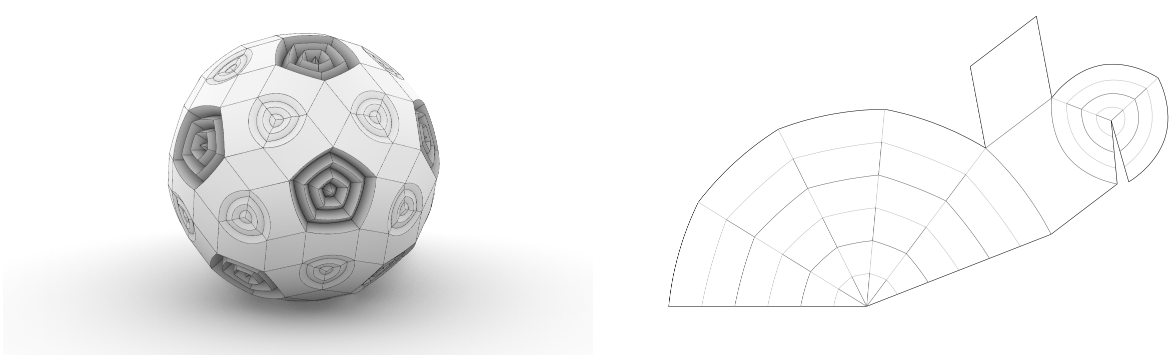

Figure 1: *A folded Icosidodecahedron and a part of its development (darker gray depicts the mountain, lighter gray the valley folds).*

Construction

The constuction of the curved crease folded spherical polygon P with vertices $P_1, \ldots, P_n$ can be decomposed into three steps, cf. Figure 2:

1. We first align right circular cylinders and planes along the egdes of P and unroll them to the plane.
2. We compute the crease curves between the cylinders and cones with an appropriate apex.
3. We can add further creases to the cone, e.g., by reflecting the cones at planes.

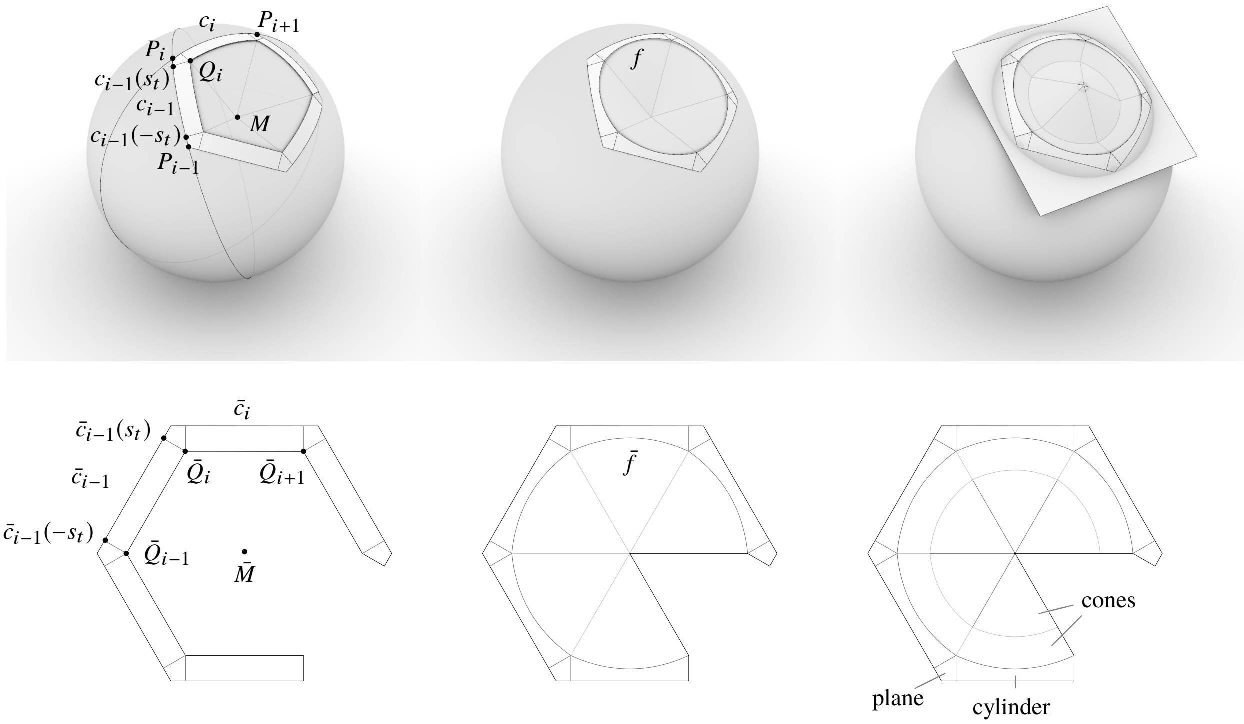

Figure 2: *Illustration of the proposed three steps in the construction of a folded spherical regular polygon.*

Parametrization

Step 1: Constructing the cylinders

We describe the method for a single regular polygon P. We parametrize P's edges P_iP_{i+1}, that is, the great circles of the underlying (unit) sphere S by $c_i(s)$, where we choose w.l.o.g. $s \in [-s_T, s_T]$ to be the arc length with $c_i(-s_T) = P_{i-1}$ and $c_i(s_T) = P_i$. We then align cylindrical patches that are tangential to S and pointing towards the interior of P along the curves c_i. At each vertex P_i, we choose a starting parameter $0 < s_t < s_T$ and determine the intersection point Q_i of the two corresponding rulings at $c_{i-1}(s_t)$ and $c_i(-s_t)$ and replace the surface between those rulings by a plane. We then unroll $Q = \{Q_1, \ldots, Q_n\}$ w.r.t. the composition of cylinders and planes into the plane, which results in an open polyline and denote the corresponding objects with a bar, e.g., $\bar{Q}$ and $\bar{Q}_i$: the lengths of the edges $\bar{Q}_i\bar{Q}_{i+1}$ are the lengths of the great circles between $c_i(-s_t)$ and $c_i(s_t)$ and the angle θ between two consecutive edges $\bar{Q}_{i-1}\bar{Q}_i$ and $\bar{Q}_i\bar{Q}_{i+1}$ is defined by $\cos\theta = \frac{P_{i-1}\times P_i}{|P_{i-1}\times P_i|} \cdot \frac{P_{i+1}\times P_i}{|P_{i+1}\times P_i|}$. As the open polyline $\bar{Q}$ has equal edge lengths and angles it possesses a circumcircle with center $\bar{M}$. In the next step, we want to fold the hereby constructed cylinders into cones with the same apex, so that the crease curves pass through the vertices of Q and $\bar{Q}$ resp. For that we choose $\bar{M}$ to be the developed cone's vertex and define its spatial counterpart M by solving for an interior point of S satisfying $|M - Q_i| = |\bar{M} - \bar{Q}_i|$.

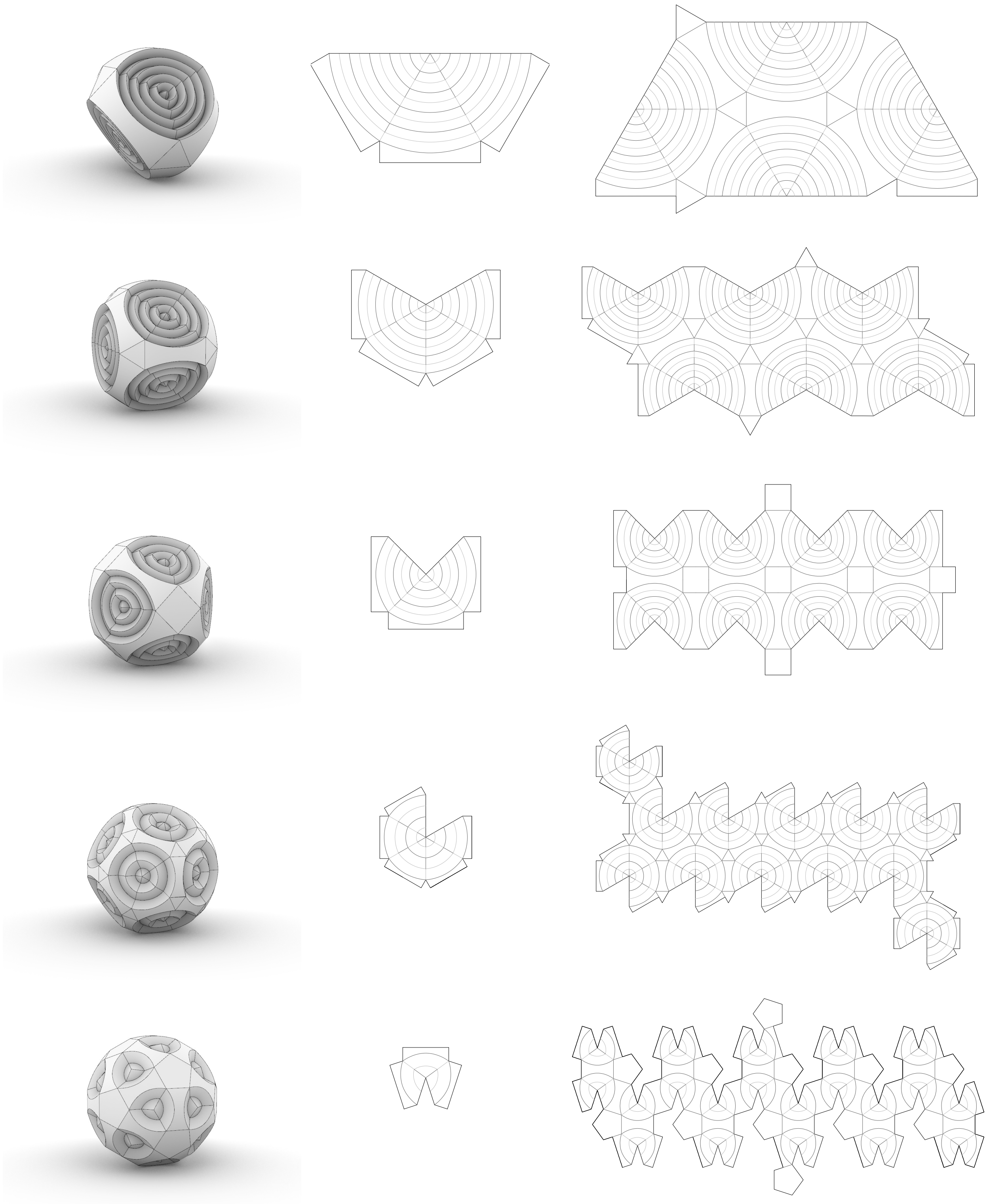

Figure 3: *Curved crease folds of the spherical Platonic solids (Tetrahedron, Cube, Octahedron, Dodecahedron and Icosahedron) and their developments.*

Step 2: Folding the cylinders into cones

As in [6] or [7], we determine the crease curve f by making the ansatz that f and its development $\bar{f}$ are parametrized by $f(s) = c(s) + l(s)d(s)$ and $\bar{f}(s) = \bar{c}(s) + l(s)\bar{d}(s)$, where c is a curve on the first developable surface, l an unknown length function and d the ruling direction. We denote their corresponding developed counterparts again with a bar. The length function l of a fold into a cone with vertex M can be computed from the isometry condition

$$|c(s) + l(s)d(s) - M| = |\bar{c}(s) + l(s)\bar{d}(s) - \bar{M}| \quad \Longrightarrow \quad l(s) = \frac{1}{2}\frac{|\bar{c}(s) - \bar{M}|^2 - |c(s) - M|^2}{(c(s) - M) \cdot d(s) - \left(\bar{c}(s) - \bar{M}\right) \cdot \bar{d}(s)}. \quad (1)$$

We can choose a coordinate system so that $c(s) = (\sin s, 0, \cos s)$, $d = (0, 1, 0)$, $\bar{c}(s) = (s, 0)$ and $\bar{d} = (0, 1)$. Note that for small values of $s_T - s_t$ the length l might become negative and thus f oversteps c. This can be adjusted by choosing a smaller parameter s_t.

Step 3: Adding further creases

These hereby constructed cones can either be trimmed or folded. In our examples we reflect the cones along planes that are perpendicular to a normal of P. The crease curves can also be computed by Equation (1) by setting $c(s) = f(s)$ to be the previous fold, $d(s) = \frac{M - f(s)}{|M - f(s)|}$ and M to be the reflected cone's apex.

Remarks

This method can also be applied to more general spherical Polygons as long as their developed polylines $\bar{Q}$ have a circumcircle. However, the computation does not prevent self-intersetion of the involved surfaces.

Figure 3 illustrates this method on Platonic solids, which can be built without additional cuts from one piece of paper.

Acknowledgements

The author thanks Helmut Pottmann for the inspiration and gratefully acknowledges the support of the SFB-Transregio programme Geometry and Discretization (Austrian Science Fund grant no. I 2978).

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