Lehmer's Dance – A Lecture Performance

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Abstract

We describe our experiences with the development, preparation, and execution of a lecture performance that combines mathematics, dance, sound, and live visuals. We present some plans for future projects that combine choreography and mathematics, also in an educational setting.

Figure 1: *Cropped stills from video recording of Lehmer's Dance (credit: Collin Wagenmakers)*

Van Berkel (choreographer) and Verhoeff (mathematician) started cooperating on *Lehmer's Dance* through *Mingler* [9], a network in the Netherlands aimed at connecting academics and artists. *Lehmer's Dance* is a 45-minute lecture performance that premiered on 2 February 2019 [5, 6, with video] at the 25th anniversary edition of the National Math Days ("Nationale Wiskunde Dagen"), attended by some 900 math teachers in the Netherlands. *Lehmer's Dance* was performed a second time on 23 April 2019 at the 55th Dutch Mathematical Congress. The nature and purpose of a lecture performance is explained well in [2] (also see [4]):

> "Lecture performances incorporate elements of both the academic lecture and of artistic performance. They function simultaneously as meta-lectures and as meta-performances, and as such challenge established ideas about the production of knowledge and meaning in each of the forms to which they refer. . . . As a hybrid format, the lecture performance always participates in more than one context.'

Lehmer's Dance explains Lehmer's conjecture supported by movement, live visuals, and sound. It also reflects on the applied choreographic method, and it contains more autonomous sections in which the permutations are artistically interpreted through sound, movement, and visuals. The lecture performance closes with an audience participation.

Verhoeff's aims with *Lehmer's Dance* include showing that math is alive through the use of contemporary and accessible math, and that embodiment can be helpful in communicating and understanding mathematics. Van Berkel' goals include the application of choreographic scores to a new intriguing conceptual domain.

Mathematical Details

Lehmer's Dance—the lecture performance is designed around a conjecture of D. H. Lehmer from 1965 concerning the following combinatorial problem [3].

> Given is a sequence of objects, some of which may be indistinguishable, for instance, because they have the same 'color'. The goal is to present *all* permutations (rearrangements) of this sequence, one by one, such that the transition from one permutation to the next involves a swap of two *distinguishable neighboring* objects, while avoiding duplication of permutations.

Lehmer knew that duplications cannot always be avoided (try for instance the sequence $AABB$), but he conjectured that the presentation can always be accomplished with a restricted type of duplications, where the same two neighbors swap twice in succession. He called such duplications *spurs*.

Verhoeff got interested in this problem in the context of letting a team of six participants at the 2009 International Mathematical Olympiad do something mathematically intriguing on stage during the opening ceremony [7]. At the time, he only knew about the special case where all objects are distinguishable (in which case there is a nice recursive algorithm to do the presentation). This problem was partly solved in the 1980s and 90s. In 2015, Verhoeff proved Lehmer's conjecture in the binary case, involving two 'colors' [8].

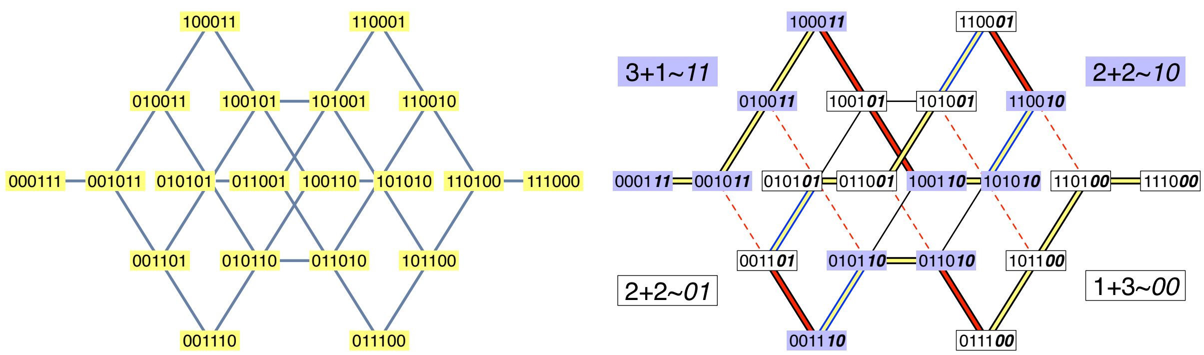

Figure 2: *Neighbor-swap graph for* $000111 = 0^3 1^3 = 3 + 3$ *(left), decomposition and integration (right)*

Two elegant insights underly that proof. Firstly, the tips of the spurs can be characterized as the so-called *stutter permutations*, where objects appear paired by 'color' (except for an unpaired object on the right, if their number is odd). For instance, $AABB$ and $BBAAC$ are stutter permutations. Stutter permutations are only present when at most one 'color' appears an odd number of times. The problem can now be reformulated as finding a Hamiltonian path in the neighbor-swap graph (see Fig. 2, left) *without* stutter permutations. The stutter permutations can be incorporated afterwards via spurs.

Secondly, in the recursive construction of such a path, it turns out that two paths for a pair of subgraphs *with* stutter permutations can be combined nicely to form one path *without* spurs. Fig. 2 (right) shows how the neighbor-swap graph for 000111 (abbreviated as $3 + 3$) decomposes into one $3 + 1$ (ending in 11), two $2 + 2$'s (ending in 01 and 10), and one $1 + 3$ (ending in 00). The paths for the two $2 + 2$s have two spurs each, and the four paths for the parts can be integrated into one path for $3 + 3$, without spurs.

Choreographic Details

There are several correspondences between mathematics and choreography. The first correspondence is structure—similar to language and music. Inherent to the notion of structure is the need for discovering and creating patterns. Choreographers often use spatial patterns (Merce Cunningham and Ana Teresa de Keersmaeker), visual patterns (Wiliam Forsythe), or musical patterns (Jonathan Burrows) [1]. There are also many examples of choreographies that use mathematical principles to lesser or greater extent. 'Accumulation' (1971) by Trisha Brown is an early postmodern choreography that revolves around minimalism with basic mathematical rules: a simple gestural phrase becomes increasingly complex through the use of repetition, accumulation, and multiplication.

Secondly, mathematicians and choreographers theorize space. There are several notation systems to describe how movement unfolds in space and time. One of the most widely used notation systems was created by Rudolf Laban (1879–1958). In order to describe space, Laban used geometry to differentiate the spatial parameters of a movement. He created a number of movement scales (comparable to musical scales) with the corners of some of the Platonic solids (tetrahedron, octahedron, and icosahedron). These scales revolve around proximity and 'minimal change'. The work of postmodern and contemporary choreographers is strongly influenced by these scales, such as 'improvisation technologies' by William Forsythe. It is currently applied not only to choreography and theater, but also in anthropology and robotics.

Thirdly, the notion of 'system' is a correspondence between mathematics and choreography. Some choreographic methods are aimed at the creation of 'scores': a set of instructions in which the performer's own interpretation plays a big role. The creation of a score is a systematic collection of elements that consequently allows the interpreter to engage with it playfully: 'Scores make one not reproduce the image one has of performing something, but going back to the score one can always retap into the conditions of the movement production.' (Jonathan Burrows).

Choreographic Method In order to create choreographic 'scores' based on Lehmer's conjecture (see [6] for the scores), Van Berkel began with the most simple way to explain the rules:

- There need to be distinguishable objects; there can also be indistinguishable objects.
- Objects are placed in a linear order (next to each other).
- Only distinguishable neighboring objects can swap places.

The rules in Lehmer's conjecture specify what and how to swap, but do not specify the nature of the objects involved. The swaps can be performed by humans, static objects such as chairs, moving objects such as robotic vacuum cleaners, or immaterial objects such as light beams or tones. Also the nature of the swaps is not specified: parameters such as direction, distance, and time can be varied.

For *Lehmer's Dance*, the rules of the conjecture have been applied to image, sound, and movement. The lecture performance makes use of sequences that differ in composition (2 + 1, 2 + 2, 3 + 3, 2 + 2 + 1). as well as different forms for the same composition. Each swap permutes the sequence and takes place on stage, in the auditorium, on the projection screen, or in the soundscape. In order to work with entities that are distinguishable as well as indistinguishable, we have used four properties: form (visuals), color (visuals and dance), pitch (sound), and movement quality. By using a live camera, we could visualize the permutations with various small objects such as walnuts, bolt nuts, and sugar cubes. In the audience participation at the end, we split the audience into four groups to do a 2 + 2, where sitting versus standing was the distinguishing property. A swap then involves two neighboring groups changing state simultaneously.

In order to give some concrete examples of the choreographic working method, we will discuss a cyclic 2 + 2 + 1, which involves 30 permutations and 2 spurs, repeated 5 times. Each permutation consists of five objects, of three distinguishable kinds. The distinction between the three kinds of objects was firstly made with colored sweaters: dark blue, orange, and green. We then choreographed the permutations, as follows.

1. *Listening, sensing* – The dancers stand in a line and permute with a focus on awareness and presence: what do you see, what do you hear, what do you smell? This 'simplest' way of performing the permutations introduces the notion of ecology: every swap is a change in the system as a whole.
2. *Move the line* – The dancers change the orientation of the line.

Besides spatial parameters, we added the use of body parts, form, duration, traveling, and level change. Each dancer made two small 'archives' of eight movements with which we created the sections described below. We then proceeded by considering the visualization of all permutations within a 2 + 2 + 1 as a matrix that can be read horizontally (see Steps 1 and 2) and vertically (see following steps). By counting the number of rows in which the dancer stays in the same position, we created a system with various durations. One dancer would for instance count durations of 2, 4, and 7, while another would count 3, 1, and 5.

3. *Same color, same movement* – Indistinguishable dancers do the same movement as they permute, which means that the audience recognizes three distinct kinds of movements in the 2 + 2 + 1.
4. *Same movement, different place* – Indistinguishable dancers do the same movement on the counts they have in common. But these movements do not happen simultaneously, because they are located in different places in the matrix.
5. *Break the line but keep the swaps* – The dancers use the whole stage with movements from their own archives, yet honoring the spatial rule of the swaps.
6. Etcetera. . .

The creation and performance of these choreographic rules within the mathematical rules resembles a playground that revolves around the combination of precision and freedom.

Future Plans

It has been an exciting cooperation where we learned a lot about each other's discipline. Lehmer's conjecture offered a rich set of concepts that are both mathematically and choreographically interesting. *Lehmer's Dance* has explored only a few possibilities. We are looking for ways of extending the project, and reworking the performance into something that can be used in education. It is also our intention to consider other minimal change algorithms, such as Gray code and prefix rotations.

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