Generalized Julia Sets: from Cantor Bouquet to Cantor Cheese

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Abstract

The complex quadratic family $z \mapsto z^2 + c$ is well known to give rise to aesthetic images and renderings of *Julia sets* — for a given complex number c, the Julia set is the boundary of the set of points z that escape to infinity under iteration. There exists a unique *critical value* c, which is the image of the critical point $z = 0$. If iterates of c remain bounded then the Julia set is connected, typically a set of closed (Jordan) curves, and if its iterations diverge to infinity then the Julia set is a Cantor set. Here, we consider a non-analytic perturbation of the complex quadratic family and its *generalized Julia sets*, which are defined in the same way. There are now infinitely many critical values that lie on a circle. Consequently, some critical values may remain bounded under iteration, while others do not. Accordingly, the (generalized) Julia set can be more complicated, leading to new intriguing mathematical objects that may provide novel inspirations for art.

Introduction

One of the paradigm examples in dynamical systems theory is the complex quadratic family [4] defined as

$$z \mapsto z^2 + c,$$

for $z \in \mathbb{C}$ and parameter $c \in \mathbb{C}$. This map generates a dynamical system for any fixed c by iteration: for an arbitrary choice of initial point $z = z_0 \in \mathbb{C}$, its orbit is the infinite sequence

$$\{z_0, z_1, z_2, \cdots\}, \text{ with } z_i \in \mathbb{C} \text{ and } z_{i+1} = z_i^2 + c, \text{ for all } i \in \mathbb{N}.$$

In polar coordinates, $z = re^{\vartheta\, i}$, the complex quadratic family becomes

$$re^{\vartheta\, i} \mapsto r^2\, e^{2\,\vartheta\, i} + c,$$

which shows that the map involves angle doubling (due to the z^2 term). The dynamics of the complex quadratic map for a given c is primarily determined by whether and which points in the complex plane remain bounded under iteration. The boundary of this set of bounded orbits is called the Julia set. It is a special property of the complex quadratic map, called the *fundamental dichotomy*, that its Julia set is connected if, and only if, the orbit of the *critical value* c is bounded; otherwise, if the orbit of c goes to ∞, the Julia set is always a Cantor set, that is, it is totally disconnected [3].

In this paper, we consider the perturbation or generalization of the complex quadratic map given by

$$f_\lambda(z) = \left(1 - \lambda + \lambda\, |z|^2\right) \left(\frac{z}{|z|}\right)^2 + c. \tag{1}$$

Here, the parameter $c \in \mathbb{C}$ is as in the complex quadratic map, but there is now an additional (real) parameter $\lambda \in [0, 1]$ that modifies the quadratic term. This family of maps was motivated by the map introduced in [2] with $c = 1$ in the context of higher-dimensional (wild) chaotic dynamics in a Lorenz-like vector field in $\mathbb{R}^5$; see also the geometric studies in [6, 7, 12] of this type of chaotic dynamics of the map (1). As for the quadratic map, it is easier to understand the action of the map (1) when it is written in polar coordinates, namely, as

$$re^{\vartheta\, i} \mapsto \left(1 - \lambda + \lambda\, r^2\right)\, e^{2\,\vartheta\, i} + c.$$

This form shows that the map (1) is essentially like the complex quadratic map, still doubling the angle ϑ, but now the radial term is $1 - \lambda + \lambda\, r^2$, which differs from r^2 if, and only if, $\lambda \neq 1$. Note that the map (1) is not defined for $z = 0$, but considering $r = 0$ gives the set $J_1 := \{(1 - \lambda)\, e^{2\,\vartheta\, i} + c \mid \vartheta \in (-\pi, \pi]\}$ as the 'image' of $z = 0$; note that J_1 is a circle of radius $1 - \lambda$ centered around c. (The notation J_1 comes from the literature on non-invertible maps and was also used in our earlier work [6, 7, 8, 9, 12].) Just as c is the critical value for the complex quadratic map, now J_1 plays the role of critical values for the map (1). By this we mean that the properties of the generalized Julia set of the map (1) are determined by the fate of the orbits starting on the critical circle J_1; precise details will be presented in the next section.

In this paper, we discuss some of the specific findings reported in [8, 9] with focus on the mathematical and aesthetic properties of the generalized Julia sets of (1). These are again defined as the boundary of the set of points that escape to infinity (for fixed c and λ), and they can be computed in exactly the same way as for the quadratic map, namely by coloring the complex plane according to the fate of initial conditions under iteration. For the images presented here, points in the complex plane are colored yellow if their associated orbits remain bounded; otherwise, we color them blue. More specifically, we consider a grid of 1000×1000 initial conditions $z_0 = x_0 + y_0\, i$ in the range $-1.2 \leq x_0, y_0 \leq 1.2$ and decide the color based on up to $N = 500$ iterations. During the iteration, if we find that $|z_i| \geq 2.4$ then the initial condition is colored blue; on the other hand, if $|z_N| < 2.4$ then the initial condition is colored yellow. This method of distinguishing initial conditions can be varied in many ways to generate aesthetically pleasing images; in particular, one can define a color gradient that represents the rate at which orbits diverge to ∞, or converge to a finite attractor (if it exists).

Generalized Julia Sets with Novel Geometric Properties

Figure 1 shows generalized Julia sets of the map (1) when $c = 0.1$ (top row) and $c = 0.28$ (bottom row). The case shown in panel (a) is representative when the orbit of any point on J_1 converges to an (exponentially) attracting periodic point. Just as for the complex quadratic map, the generalized Julia set is then a set of so-called Jordan curves that separate the basins of attraction of the attracting periodic point from the points that go off to infinity; in Figure 1(a) there is an attracting fixed point and the Julia set is a single Jordan curve. Similarly, if none of the points in J_1 remain bounded then the generalized Julia set is a Cantor set, that is, totally disconnected (not shown).

New types of generalized Julia sets arise for the ambivalent case where some points on J_1 escape to infinity and other points on J_1 remain bounded. Figure 1(b) shows an example for the particular case where the critical point $z = 0$ lies *inside* J_1. The generalized Julia now forms a *Cantor bouquet*, which is an infinite union of arcs, connected in a single so-called *explosion point*, such that the end points of the arcs are dense. The Cantor bouquet as a Julia set has been discovered before, namely, in the complex exponential family $z \mapsto \lambda\, e^z$; there, it exists when $\lambda < e^{-1}$, but then the explosion point lies at infinity [1, 5, 10, 11]. Panels (c) and (d) of Figure 1 show two additional ambivalent cases. As in panel (b), the generalized Julia sets in panels (c) and (d) are connected, but they each have fundamentally different geometric properties. For both of these cases, the critical point $z = 0$ lies *outside* J_1 and the points on J_1 that escape to infinity form a set of curve segments. At the same time, J_1 contains infinitely many points that remain bounded. The difference is that in panel (c) the map (1) does not have a finite attractor, while in panel (d) it does. When there is no finite attractor, the generalized Julia set is a *critically connected Cantor set*, which is characterized by the property that it is connected at a countable dense set of points and it contains Jordan curves that bound open regions of points that escape to ∞. In Figure 1(d), a finite attractor exists, so that there also exist open regions of

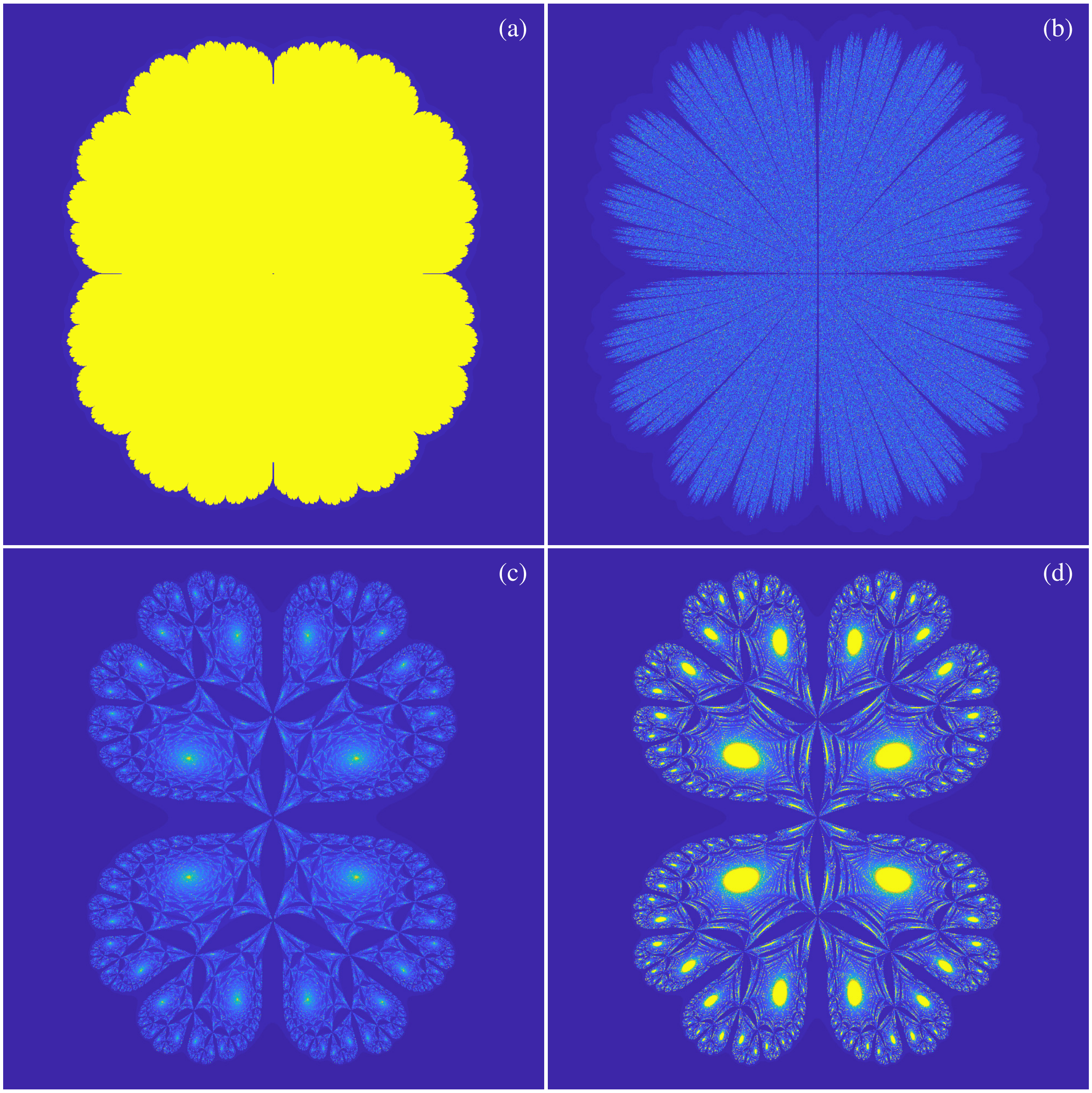

Figure 1: *Generalized Julia sets (boundary of the blue region) of the map* (1)*. Panel (a) for* $c = 0.1$ *and* $\lambda = 0.78$ *is a Jordan curve. Panel (b) for* $c = 0.1$ *and* $\lambda = 0.6$ *is a Cantor bouquet. Panel (c) for* $c = 0.28$ *and* $\lambda = 0.91$ *is a critically connected Cantor set. Panel (d) for* $c = 0.28$ *and* $\lambda = 0.89$ *is a Cantor cheese.*

points that converge to this attractor; consequently, we color these yellow. We call this generalized Julia set a *Cantor cheese*.

Conclusions

The map (1) exhibits more complicated generalized Julia sets than are known for the complex quadratic map. This is due to the new and third possibility that some points on the critical circle J_1 escape to infinity, while others do not. The generalized Julia sets for this ambivalent case come in a number of different variants, and we showed here a Cantor bouquet, a critically connected Cantor set, and a Cantor cheese. There are more cases, but it is not yet clear whether all cases have been discovered; see [8, 9] for more details.

From an artistic point of view, these new types of generalized Julia sets offer opportunities for creating novel, intriguing and aesthetic renderings. While they are just as straightforward to generate, generalized Julia sets are fundamentally different from well-known images of Julia sets for complex quadratic maps and other complex analytic maps. Hence, the map presented here can be seen as a new playground that may inspire the mathematical arts community.

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