Impossible Pictures: When Art Helps Math Education

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Abstract

The paper talks about the use of the impossible figures in science education discovering, in particular, the great artistic and communicative value of the artworks of Oscar Reutesvärd, that long before Escher and Penrose, had drawn impossible triangles and stairs using the "Japanese" perspective. In the last part of paper we analysed the links between art and impossible figures showing how over the centuries their communicative power has been used by W. Hogarth, and Op Art, movement which tried to involve the observer deceiving his eyes and his brain.

Introduction

"*Everybody knows that a thing is inexplicable until some unknowing fool comes along and explains it*". So Einstein said about impossible situations that often need to be addressed when we talk about science at any level, especially in teaching. In my opinion, assigning questions to students that lead to impossible situations is a very powerful way to help them to develop key skills. Impossible situations have the great benefit to implicitly require the development of metacognitive reasoning on the matter, not only a trivial application of the rules, of which it is not able to offer a justification.

If you ask to a student in the first year of high school to build an equilateral triangle with defined size, he will build it without any problem using ruler and compass. If instead we propose to the student the realization of a triangle with internal angles that measure 75°, 45° and 65°, he may (or better, should) be able to assert that it is impossible to build this triangle and, if instead he tried to build it accurately using ruler and goniometer, you will face an impossible situation which leads him to reflect on the nature of the figure and its links to the values of the internal angles. At the same time, if you ask to a high school student to solve the equation $2x = 46$, he will be able, perhaps through the mechanical process of dividing both sides by 2, to solve the equation. If you require to solve the equation $0x = 16$, education research [1] tells us that some students propose 0 as a solution, others 16, and only a fraction (under 50%) immediately answers that is an impossible equation. The reasons for these difficulties [1] are to be found in the inability to skip from "natural" language to symbolic language, i.e., the inability to translate the relationships between the variables in the relationship between symbols and back again. In fact, to solve the second equation you must use metacognitive reasoning that leads to the translation from the algebraic register to the numerical register (*What number multiplied by 0, results in 16? None, then the equation is impossible*), or even using only mechanical rules which, however, add the reasoning that it is impossible to divide by zero and so that the equation is impossible.

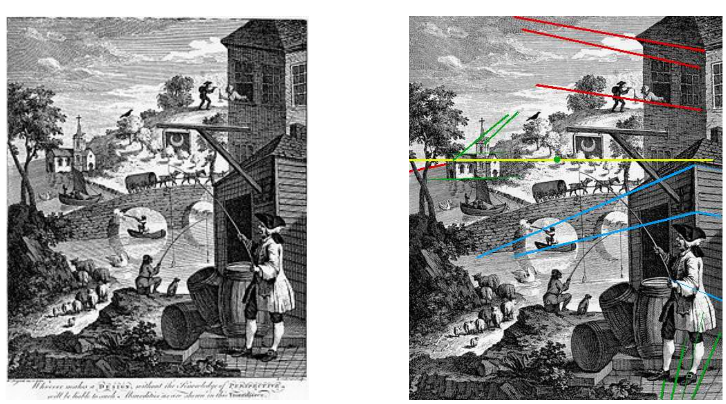
From these two examples it is easily seen that when we ask to solve an exercise that will be impossible, students must perform a sort of metacognitive reasoning that helps them to realize what they really understood. In particular, this avoids the continued use of structures that have been consolidated by only repeating mechanical operations that can be used to fix concepts, but that do not help to develop specific skills. In my opinion, this is a very useful way to develop basic skills in high school and, as we will see later in the paper, this can also be developed using artistic and graphic works that will increase the interdisciplinary nature of the teaching proposal.

Oscar Reutersvärd: the father of impossible pictures

An impossible object is a type of optical illusion. It consists of a two-dimensional figure which is instantly and subconsciously interpreted by the visual system as representing a projection of a three-dimensional object. In most cases, the impossibility becomes apparent after viewing the figure for a few seconds. However, the initial impression of a 3D object remains even after it has been contradicted. There are also more subtle examples of impossible objects where the impossibility does not become apparent spontaneously and it is necessary to consciously examine the geometry of the implied object to determine that it is impossible. The father of impossible figures is commonly identified as the Swedish artist Oscar Reutersvärd (1915-2002). Reutersvärd's originality appeared early in his career at the age of 18. In 1934, the school student created a figure, the "Impossible triangle", composed of a series of cubes in perspective, called by Reutersvärd himself *Opus 1*.

Reutersvärd's artworks are based on a perspective trick called "Japanese perspective": an object, or a series of objects, are seen simultaneously in multiple aspects (at least 2, sometimes 3) under different points of view, but in a way that there is a "weld" between the resulting figures, in a general solution that cannot exist, and so is realistically absurd [2]. Reutersvärd achieved more than 2500 artworks using the purity of essential figures, all geometric pictures. Only the latest works are "contaminated" by the use of watercolours. He believed that the aesthetic beauty of his creations consisted in the "impossible figures" themselves, not in the magic that could arise from them and that instead somehow fascinated and inspired the works of the other great artist who made use of impossible figures: Escher. Reutersvärd contends that he did not need it, relying only upon the pure figure. For example, like Escher, Reutersvärd also transformed some of his "impossible figures" into stairs, but he never felt the need to let it be climbed by monks or covered by water in perpetual descent; his work is always and only limited to implicitly suggesting to viewers to approach it with imagination! The particular perspective of Reutersvärd's works, with their declared intention to be false, represents a revolt towards the Renaissance scientific perspective and the implicit anthropocentric view [3]. To express his conception of the world Reutersvärd choose this particular perspective because in it, all parallel lines remain parallel and do not converge visually at any point. In this way, we create numerous angles that generate different geometric shapes that join with each other giving rise to an object that does not exist in reality but that our brain has to interpret to be able to extrapolate the realistic side. In the last artworks Reutersvärd experimented with new graphic techniques linked to new technologies and the improvisation of figures made impossible when listening to Bach Fugues [4].

If in Reutersvärd we recognize the conscious use of the "Japanese Perspective" and the remarkable creativity inside the impossible figures, it is important to remember that other artists used impossible figures to create forced perspective, such as the painter William Hogarth (1697-1774) in his famous *False perspective* of 1754 (Figure 1a).



(a) (b)

Figure 1: *(a) False perspective, W. Hogarth - 1754, (b) and with visual lines (added by the author)*

Hogarth uses fishing line and the lighting of a pipe to make a paradoxical scene that just using the brain can admit several perspective deceptions that the eye alone cannot fix (Figure 1b). The work of Hogarth is considered important in the history of design and in the development of modern graphic communication languages. His artworks are also present in many science textbooks to illustrate the deceptions of optical illusions, demonstrate the importance of knowing how to make good use of representational visual tools to communicate correct scientific points. Among the other artists who have used impossible pictures are also the Italian artists Andrea Pozzo, and his spectacular work in the Saint Ignazio Church in Rome, Bramante, in his particular perspective paint in Saint Satiro Church in Milan, not to mention Dalì, Arcimboldo, Magritte and many others [5].

Impossible pictures in the didactic of geometry

In this section, I talk about the use of impossible figures to develop scientific skills in high school. Impossible figures represent a unique meeting point between the mathematical world and the artistic one and recall the medieval figurative world, in which proliferated multiple spatial reading and relief images in which the characters penetrate into each other. The impossibility of Reutersvärd's artworks is not always obvious; not everyone immediately perceives the paradox of some of them, as the human eye tends to see in a two-dimensional depiction of an object that is still three-dimensional. In Italy, there are some educational courses based on Reutersvärd's works for primary school realized by B. D'Amore, one of the most important world experts on Reutersvärd's artworks[1].

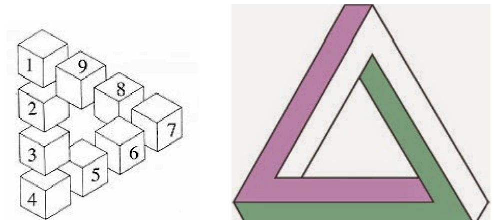
The use of art in mathematics education, in particular in geometry, is recognized as an added value by international research [6] and by national and international projects [7] because art and mathematics are characterized by an analogue inductive and visual type of thinking/learning. Both use metaphorical and symbolic languages to explore the world behind the world and arrive at the meaning of what we perceive every day. Using art in mathematics education allows students to acquire a conceptual aesthetic: in addition to the mechanical memorization of formulas, images help students to give meaning to what those formulas represent. Impossible figures give the observer a means not only to consider spatial geometry, but also the logic of space. Usually my geometry lectures include, in the first class of high school (Liceo Scientifico), some lesson on the important link between math and art in addition to the participation in the national competition of excellence "Adotta Scienza e Arte nella tua classe" [7] since the 2012-2013 school year. During the 2018-19 school year, I dedicated 4 lessons to impossible pictures and talked about the links between mathematics and art. The outline of the lesson is usually this:

- An aphorism of a scientist or artist on the links between art and science is read and discussed;
- Some impossible figures are shown;
- The students discuss the mathematical and physical aspects that imply the impossibility and their interdisciplinary links with other subjects.

In the following, I will briefly describe, valuing their interdisciplinary qualities, the four lessons.

Lesson 1. We discussed H. Matisse's aphorism "*Seeing is already in itself a creative act*" and the different meaning of seeing an object (or an artwork) and observing it. To address the distinction between these two activities, we analyse the artworks *Opus 1* (1934), the first intentional impossible object of Reutersvärd, and the Penrose triangle. In 1934, Reutersvärd, bored during a Latin lesson, began to scribble a sort of six-pointed star, and encircled it with 3-dimensional cubes. If we number the cubes from 1 to 9 (Figure 2a) and we observe only the cubes from 1 to 7 (without the 8th and 9th), the perspective is correct and they have the direction from left to right of the beholder; if you observe the cubes from 4 to 9 (excluding the 2nd and 3rd), the outlook is still correct, but they have the direction from right to left of the beholder; you can also proceed removing 5 and 6, still getting a proper perspective [4]. The goal is try to reconcile all these partial versions in one piece, in a single figure: you have more points of failure which transform the locally correct figure in a globally impossible one. This is an aspect that one good student with a geometric vision should detect, and that helps him to develop basic skills.

[1] http://www.formath.it/divulgazione-scientifica/le-figure-impossibili-di-oscar-reutersvard/.



(a) (b)

Figure 2: *(a) Opus 1 (with numbered cubes) by O. Reutersvärd, 1934, (b) Penrose Triangle.*

Closely linked, and very similar, to the Reutersvärd's *Opus 1* is the *tribar* or *Penrose triangle* (Figure 2b), presented by the Penroses, father (Lionel Sharples) and son (Roger), in their paper [8] published in 1958 and created by them in a totally independent way from Reutersvärd. They were inspired by Escher's work and Reutersvärd's work was, at the time, unknown to the Penroses. Roger Penrose only discovered Reutersvärd's work in 1984. The tribar appears as a solid object, made of three straight beams of square cross-section which meet pairwise at right angles at the vertices of the triangle they form. The beams may be broken, forming cubes or cuboids. This combination of properties cannot be realized by any 3-dimensional object in ordinary Euclidean space. Such an object can exist in certain Euclidean 3-manifolds. There also exist some 3-dimensional solid shapes each of which, viewed in a certain angle, appear the same as the 2-dimensional depiction of the Penrose triangle (Figure 3).

Figure 3: *Penrose triangle sculpture in Perth (Australia).*

During this lesson I propose to the students a 3D construction of the Penrose triangle (Figure 4) that can be created in an almost amazing way through simple perspective trickery. As for the building process, it is enough to have paper and scissors, to print and cut out a template available online[2] and then realize the assembly according to the related instructions. The triangle is unreal, but observing it from a certain point of view can lead our mind to see it, confusing us between two different levels of perception!

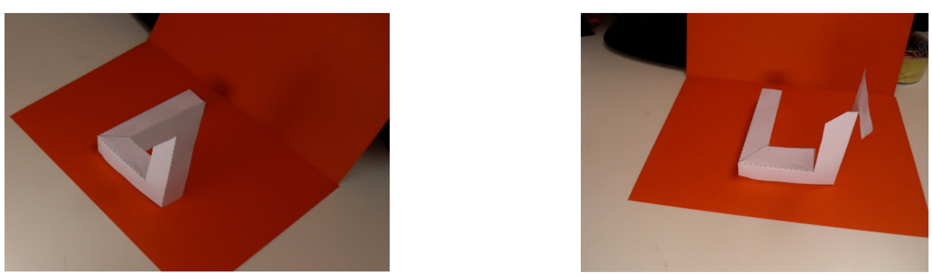


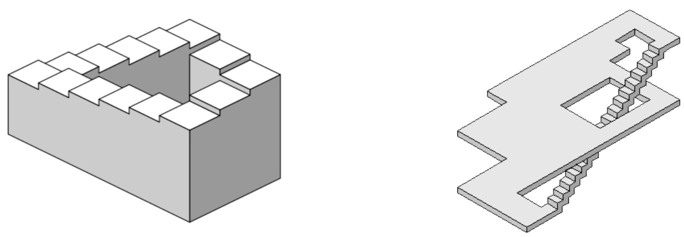
Figure 4: *Students construction of a Penrose triangle (different points of view).*

This work is the starting point for a reflection on astronomy and on the conformation of constellations, which are what we can see from our point of view on the Earth: the stars occupy a very extended three-

[2] http://www.stem2.org/je/tribar.pdf.

dimensional space. If we were go away from the Earth, the constellations as we know them would disappear, because the stars would assume a different conformation.

Lesson 2. We discussed M.C. Escher's aphorism "*The drawing is illusion: it suggests three dimensions although on paper there are only two*" and how art can transmit abstract mathematical and physics concepts, even very complex ones, in a visual form by appealing to visual paradoxes. In this lesson, we analysed some impossible pictures by Penrose, Reutersvärd and Escher to understand how their impossibility can be explained in a physical and mathematical way. In 1937, Reutersvärd created his first impossible stairs, much earlier than Escher and Penrose. Inspired by Mozart's compositional method, described as a "creative automatism", Reutersvärd began drawing a series of impossible staircases in the same "unconscious and automatic way" [4], [5] during a trip from Stockholm to Paris (Figure 5b). He did not realize, while drawing, that his figure was a continuous flight of stairs, but the process allowed him to draw ever more complex drawings, step by step.



(a) (b)

Figure 5: *(a) Penrose impossible staircase, (b) Reutersvärd impossible staircase.*

In an independent way, R. Penrose drew his impossible staircase (Figure 5a), a variation of the tribar, that is a two-dimensional depiction of a staircase in which the stairs make four 90° turns as they ascend or descend, yet form a continuous loop, so that a person could climb them forever and never get any higher. This is clearly impossible in three dimensions. In their original article, the Penroses noticed that "*each part of the structure is acceptable as representing a flight of steps but the connexions are such that the picture, as a whole, is inconsistent: the steps continually descend in a clockwise direction*" [8]. Our visual system selects a simple interpretation based on local relationships within the figure, rather than choosing a complex, yet correct, interpretation that takes the entire figure into account. We observe that each stair that is one step clockwise from its neighbour is also one step downward, and so we perceive the staircase as eternally descending. In principle, we could instead perceive the figure correctly as depicting four sets of stairs that are discontinuous, and viewed from a unique perspective, however such a perception never occurs.

After this conversation, I presented the students a solid model of the Penrose impossible staircase (Figure 6) to show them how this three-dimensional realization of an impossible object must necessarily use perspective and constructive tricks to realize what is impossible in the Euclidean space. In the solid model, the stairs at the top of the four walls form a loop, suggesting from a particular point of view that if we continue to ascend the stairs, we will eventually come back to the starting point, which is impossible since ascending the stairs should bring us always to a higher position. I showed another view of the object, from which we can see that the stairs on the left rear wall are not normal. Note that in this realization, all faces are planar and the structures that look connected are actually connected. Then students analysed, more consciously, Escher artworks such as *Waterfall*, *Ascent and Descent*, and *Belvedere*, where respectively the tribar and Penrose staircase appear, recognizing not only a geometric but also a physical impossibility that defies the laws of gravity. This is another great utility of impossible figures, thanks to which we can develop transversal key skills. In my opinion, supporting the use of impossible figures can promote inclusion and help children with learning difficulties or with space-time recognition deficiencies to understand that a single object can take different perspectives and forms depending on the points of view.

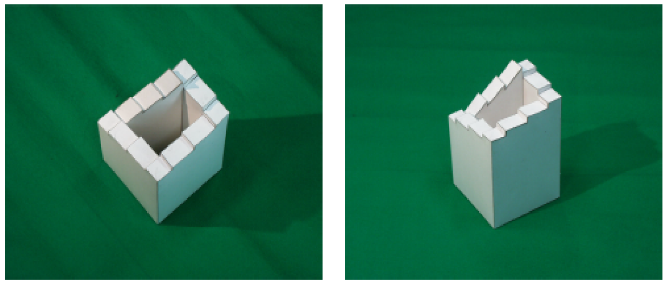
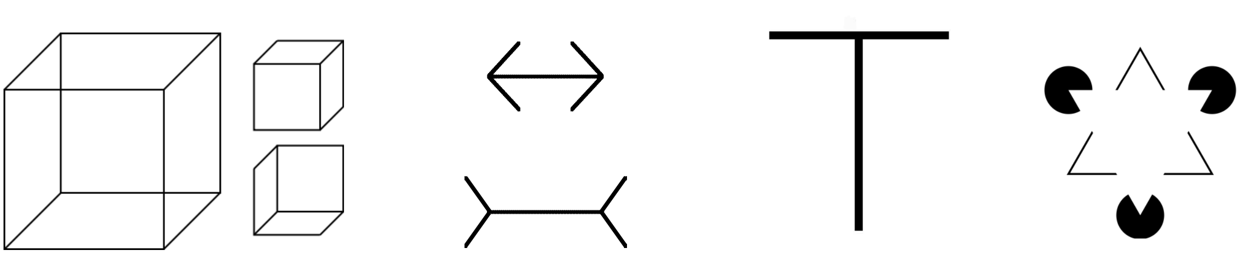


Figure 6: *A solid model realized from the Penrose impossible staircase (different points of view).*

In 1979, in fact, Gibson [9] coined the expression "Pictorial perception" to indicate the ability to see objects and scenes resulting from stimulation conditions which do not correspond at a physical level to the scene observed. Gibson spoke of a conflictual relationship on a perceptive level between the properly physical nature of an image and what can be seen in it and described it as a real paradox. From a purely descriptive point of view, the experience of pictorial perception outlined by Gibson is in some ways similar to the one seen when looking at an impossible figure, in which we see a three-dimensional object that cannot actually be such outside pictorial reality. This position is reflected in the work of other academics, such as in Pirenne [10] and in Kubovy [11], according to which the perceptive awareness of pictorial support is an essential requisite for the functioning of compensatory processes aimed at correcting perceptive distortions due to the discrepancies between the immobile geometry internal to the pictorial scene and the continuous transformations due to the changing geometry of the observation. In general, the notion of past visual experience as a factor able to modulate our perceptive experiences constitutes information for the processing of quantitative and qualitative inferences inherent to a spatial experience. The presence of a double reality, to pay more attention to one aspect than another, and control of the changing photo geometric relations in the same visual scene [12], force the observer to pay more attention to the artwork and to compare what he observes with his experiences and sensorial knowledge, helping him to bridge the gap of his possible space-time deficit.

Lesson 3. This lesson was dedicated to some optical illusions linked to the human visual system. We analysed the Necker cube, an optical illusion first published as a rhomboid in 1832 by Swiss crystallographer Louis Albert Necker, the arrows of Mueller-Lyer, the T of Pacioli (or T-illusion) and the Kanisza triangle. The Necker cube is an ambiguous line drawing. The effect is interesting because each part of the picture is ambiguous by itself, yet the human visual system picks an interpretation of each part that makes the whole consistent (Figure 7a).



(a) (b) (c) (d)

Figure 7: *(a) Necker cube - possible views, (b) Mueller-Lyer arrows, (c) T-illusion, (d) Kanizsa triangle*

In the illusion of Mueller-Lyer (Figure 7b) the same line is identified as longer or shorter depending on whether it ends with dashes facing inwards or outwards. In the T-illusion (Figure 7c), proposed by Luca Pacioli in the *De viribus quantitates*, two segments of equal length are placed one above the other to form a T, but at first glance we would say that the vertical segment is longer than the horizontal one. The students were asked to observe the two figures and say "at a glance", that is without measuring, which of the segments in each figure was the longest; then they had to confirm or modify their answers after measuring the segments of the two pictures with a ruler. The Necker cube was used to talk about the human visual system and the mechanisms related to vision that our brain controls and imposes on us. Indeed, humans do not usually see an inconsistent interpretation of the cube. The Kanisza triangle (Figure 7d), where the spatially separate fragments give the impression of a bright white triangle defined by a sharp illusory contour, and other visual illusions are useful stimuli for introducing the neural basis of perception because they hijack the visual system's innate mechanisms for interpreting the visual world under normal conditions and also to introduce the Gestalt theory. These topics are, obviously, very complex to introduce in a first high school class, but they have been developed in a qualitative way and will be elaborated during the whole course of study, also linking to other scientific subjects.

Impossible pictures and Op Art

In the fourth (and last) lesson we introduced Op Art, an artistic movement which has been influenced by and whose creators most deliberately have used optical illusions, as a virtuous ending of our first trip into the wonderful world of impossible objects. *Time Magazine* coined the term *op art* in 1964, in response to Julian Stanczak's show *Optical Paintings* at the Martha Jackson Gallery, to mean a form of abstract art that uses optical illusions. Its distinguishing feature is a strong emphasis on mathematical order. Sometimes it is accompanied by effects intended to dazzle and wrench the eye: vivid colours that generate strong afterimages when the eye shifts, optical illusions, striped and dotted patterns that torture the brain like the retinal scintillations of migraine. The artists want to get through lines placed in various modular and structural grills, effects that induce a state of perceptive instability. In this way, they stimulate the involvement of the observer [13] through a dynamic visual art that stems from a discordant figure-ground relationship. The best known method is to create effects through pattern and line: these paintings are black and white, or otherwise grisaille as in Bridget Riley's famous painting *Current* (Figure 8a). From 1967, Riley began to produce colour-based Op art (Figure 8b). However, other artists, such as Julian Stanczak and Richard Anuszkiewicz, were always interested in making colour the primary focus of their work. Often, colour works are dominated by the same concerns of figure-ground movement, but they have the added element of contrasting colours that produce different effects on the eye.

(a) (b) (c)

Figure 8: *(a) Current - 1964, (b) The Responsive Eye - 1964, (c) cover Scientific American, July 1965.*

The choice to talk about Op Art was made to repeat, once again, in the conclusion of our educational path, that the links between science and art are very strong and last for millennia, perhaps since the birth of civilization [14]. Furthermore, Op Art also allows us to return to ancient questions about art and mathematics: *To what extent is art ruled by mathematical laws? To what extent can pure mathematical structure arouse aesthetic emotions?* Clearly this choice could be criticized, but whatever one's attitude toward Op Art, there is no denying its fascination. Nor is it surprising that many Op patterns are closely related to problems of recreational mathematics. During the lesson we consider the nested and rotating squares that appear in so many Op Art paintings and fabric designs and that whirl inward on the cover of *Scientific American,* July 1965 (Figure 8c). The pattern can be interpreted as an illustration for the well-known “four-bug problem” and on this interesting mathematics problem we have read a M. Gardner paper [15] as a conclusion to the lesson and the didactical path. In the contemporary Italian school, the most difficult objective for every teacher is conquering the students’ attention in an original way this is suited to them: art is, for its natural appeal and ties with science, one of the most effective tools for math education and to start a cognitive process in the students, that can arouse a critical spirit among them and that leads them to observe nature in every aspect. This is what we hope to pursue over the years with this project on impossible pictures and, in general, on mathematics and art.

Final remarks

The contribution of the intelligence is an indispensable condition for artwork, but even more so is the presence of a conscious observer. The artist, like the scientist, does not create his work from nothing, but he realizes it according to sensations elaborated by his mind. Both are always looking for innovative ideas, to find new sources of inspiration for extraordinary creations or discoveries. The sense of sight is the one that has most contributed to the evolution of the human species and to scientific progress. In this paper we have tried to valorise also its educational value for the development of scientific skills. Probably M. Duchamp was right when said “*the painting should not be only retinal or visual; it should have to deal with the gray matter of your intellect, instead of being purely visual*”. It is my intention in the future to deepen the lessons of this didactical project, to add to the topics of the third lesson the staircase of Schröder, the Thiery module, and the figure/background Rubin effect. In particular, I want to focus my interest on these issues, the automation of vision and also towards the way that man and mind express themselves and communicate with reality and the external world; in my opinion it may be a quite original way to popularize mathematics, art and cognitive sciences.

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