Fractal Images from Multiple Inversion in Circles

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Abstract

Images resulting from multiple inversion and reflection in intersecting circles and straight lines are presented. Three circles and lines making a triangle give the well-known tilings of spherical, Euclidean or hyperbolic spaces. Four circles and lines can form a quadrilateral or a triangle with a circle around its center. Quadrilaterals give tilings of hyperbolic space or fractal tilings with a limit set that resembles generalized Koch snowflakes. A triangle with a circle results in a Poincaré disc representation of tiled hyperbolic space with a fractal covering made of small Poincaré disc representations of tiled hyperbolic space. An example is the Apollonian gasket. Other such tilings can simultaneously be decorations of hyperbolic, elliptic and Euclidean space. I am discussing an example, which is a self-similar decoration of both a sphere with icosahedral symmetry and a tiled hyperbolic space. You can create your own images and explore their geometries using public browser apps.

Introduction

Inversion in a circle is nearly the same as a mirror image at a straight line but it can magnify or reduce the image size. This gives much more diverse images. I am presenting some systematic results for multiple inversion in intersecting circles. For two circles we get a distorted rosette with dihedral symmetry and three circles give periodic decorations of elliptic, Euclidean or hyperbolic space. This is already well-known, but what do we get for four or more circles?

Iterative Mapping Procedure for Creating Symmetric Images

The color $c(\mathbf{p})$ of a pixel is simply a function of its position $\mathbf{p}$. Symmetries of an image are defined by a number n of bijective mappings M_i, where $i = 1, \ldots, n$. These typically are mirror images or inversions in circles with $M_i M_i \mathbf{p} = \mathbf{p}$. The image is symmetric with respect to these mappings if they do not change its color, $c(M_i \mathbf{p}) = c(\mathbf{p})$ for all i.

We can easily get a symmetric image for a single mapping M_i. It divides space into two regions R_i and $\bar{R}_i$ such that all points of R_i are mapped into $\bar{R}_i$ and inversely. For a mirror symmetry, they are the two half-planes on either side of the mirror axis and for an inversion in a circle they are the inside and the outside of the circle. We define a one-way map $\tilde{M}_i$, that maps all points of a mapping region R_i into the target region $\bar{R}_i$, $\tilde{M}_i \mathbf{p} = M_i \mathbf{p}$ for $\mathbf{p} \in R_i$, and does not move the points of the target region, $\tilde{M}_i \mathbf{p} = \mathbf{p}$ for $\mathbf{p} \in \bar{R}_i$. Taking the color as a function of the mapped position of a pixel $c(\tilde{M}_i \mathbf{p})$ results in a symmetric image as $\tilde{M}_i \mathbf{p} = \tilde{M}_i M_i \mathbf{p}$.

To get an image with all symmetries we compose a one-way mapping T that executes all one-way mappings M_i in some sequence. The composed mapping is repeated until its result $\mathbf{q} = T^h \mathbf{p}$ does not change anymore, $T\mathbf{q} = \mathbf{q}$. Mostly, this procedure terminates rapidly. The number of iterations h depends on the starting position $\mathbf{p}$ and the order of the execution of the one-way mappings $\tilde{M}_i$ used in T. However, for many choices of the mappings M_i the end position $\mathbf{q}$ is only a function of the starting point $\mathbf{p}$. Then it has the symmetries of all mappings, $\mathbf{q}(\mathbf{p}) = \mathbf{q}(M_i \mathbf{p})$ for all M_i. The end-point $\mathbf{q}$ lies in the intersection $\bar{R}_T = \bar{R}_1 \cap \bar{R}_2 \cap \cdots \bar{R}_n$ of the target regions of all basic mappings $\tilde{M}_i$. This is the target region of the converged iterated mapping T^h. Its complement $\bar{R}_T$ is the mapping region of T. It is the union of the mapping regions $R_T = R_1 \cup R_2 \cup \cdots R_n$ of the M_i. If R_T is multiply connected then the target region $\bar{R}_T$ is made of several

disconnected regions. There have to be lines in the mapping region R_T that separate points going to different disconnected regions. They are the limit set of the group generated by the M_i and may be simple circles or elaborate fractals. The number of iterations h diverges at these lines. You can see the trajectories of the iterated T in my browser apps[14, 15] by clicking the right mouse button on the image using the stereographic or Poincaré disc projection.

We can use the result of this procedure in various ways. To find the limit set we emphasize those pixels that need many steps to end up in the target region $\bar{R}_T$ or that never get there. For seeing the tiling structure we use different colors for pixels that used either an even or an odd number of basic mappings M_i. In addition, the colors depend on the region where the endpoint $\mathbf{q}$ lies. To create a kaleidoscopic image we take the position of the end point $\mathbf{q}$ and look up its color in an input image. These kaleidoscopic images are not only decorative, but they also show more clearly the overall symmetric structure.

For more information on the limit set of multiple inversion in non-intersecting circles see "Indra's Pearls" by C. Series, D. Wright and D. Mumford [5, 10, 9]. You can find other discussions of its fractal nature elsewhere [1, 4, 2].

Inversion in a Single Circle

The inversion in a circle of radius R with its center at position $\mathbf{c}$ maps a point at $\mathbf{p}$ to an image point $M(\mathbf{p}) = \mathbf{c} + R^2/|\mathbf{p} - \mathbf{c}|^2 \cdot (\mathbf{p} - \mathbf{c})$. This is as easy to program and evaluate as the mirror image at a straight line. It is a Möbius transformation of the complex plane combined with a complex conjugation and thus does not change the local angle of intersection between two lines. Close to the circle, inversion is the same as a mirror image at a tangent line. Looking at small regions, the inverted image is simply a translated, rotated, scaled copy of the original with reversed orientation. Overall, inversion causes distortions and bends straight lines. Repeating inversions in several circles gives a dynamical system with chaotic or fractal results similar to the dynamics creating Mandelbrot and Julia sets.
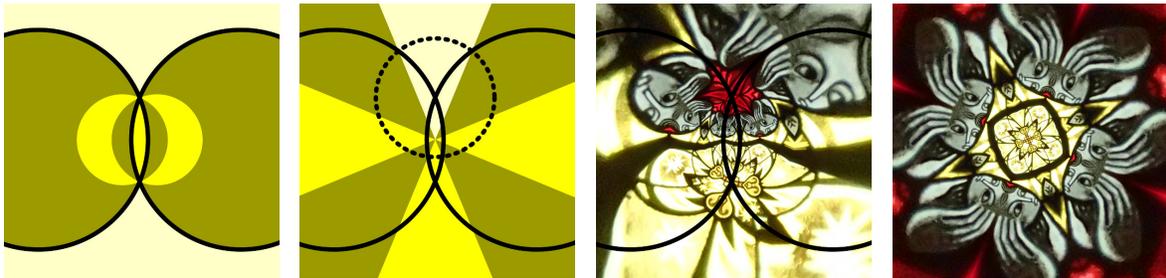
Figure 1: *Multiple inversion in two circles (solid lines). From left to right: The resulting structure, the structure after inversion in the dotted circle, a kaleidoscopic image and the same image after inversion in the dotted circle.*

Inversion in Two Circles

In Figure 1 you see the result of multiple inversion in two circles. Both one-way mappings $\tilde{M}_1$ and $\tilde{M}_2$ map the inside of the circle to the outside. Thus the outside of both circles is the target region of the converged iterated combined mapping $T^h = (\tilde{M}_2 \tilde{M}_1)^h$. It is shown in light yellow. Points that need an odd number of mappings are shown in darkened yellow and points of even number as pure yellow. You see several distorted copies of the target region, resulting from inversions. The second figure at the left shows the same result, but inverted in the dotted circle, which has its center at an intersection of the two circles. This intersection

point is mapped to infinity and the circles thus become straight lines. Inversions in the circles then become mirror images at the straight lines. The multiple mirror images make a dihedral group. At the right we see corresponding kaleidoscopic images.

We only get an image with dihedral symmetry if the intersection angle α of two circles is an integer fraction of the straight line, $\alpha = 180°/k$. Note that this is also necessary for giving the same result from multiple inversions independently of the sequence of mappings used for the composition of T. The result of multiple inversion in two circles thus depends essentially only on the integer k. Actual images only appear to be different, but they all have the same dihedral symmetry D_k.
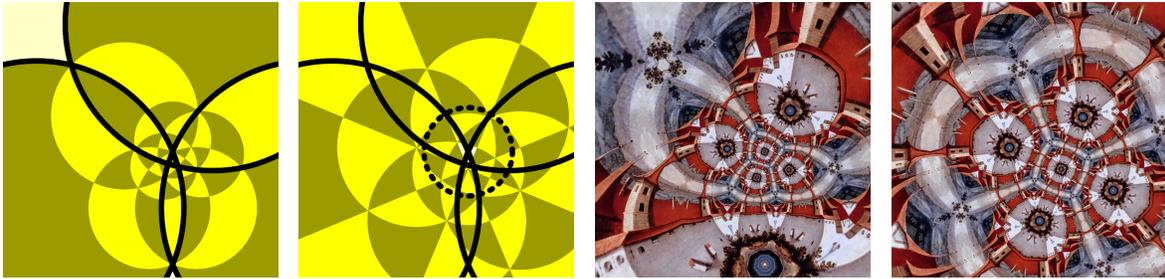
Figure 2: *As Figure 1, but for multiple inversion in three circles resulting in elliptic geometry.*

Figure 3: *As Figure 1, but for multiple inversion in three circles resulting in Euclidean geometry.*

Figure 4: *As Figure 1, but for multiple inversion in three circles resulting in hyperbolic geometry.*

Three Circles

Suppose that each circle intersects or touches the other two. The three intersection angles have to be integer fractions of $180°$, as for two circles. They can be characterized by a triplet of integers (K, N, M), which determine the angles at the intersections and the order of the corresponding dihedral groups. We can again use one-way maps that invert only the inside of the circles to their outside.

Figure 2 shows an example with intersection angles of 45°, 60° and 90° corresponding to (4,3,2). The circles are overlapping and their intersection is a triangle with these angles. Their sum is larger than 180° and thus we have elliptic geometry. Multiple inversion gives an octahedral symmetry combining D_4, D_3 and D_2. Note we can again use inversion in a circle to transform two of the circles into straight lines. This makes the symmetry evident. The third circle is inverted too, exchanging its inside with its outside. Its one-way map thus changes and maps its outside to the inside in the inverted view.

In Figure 3 the intersection angles are 90° and two times 45° resulting from (4,4,2). Their sum is 180°, giving Euclidean symmetry and thus the circles intersect in a single point. Note that the resulting image is made of circles and thus we do not have a spiral, as one might believe at first sight. Actually, this is an inverted image of a periodic Euclidean tiling.

The three circles in Figure 4 surround a triangle with angles of 36° and two times 45°. This corresponds to (4,4,5). The sum of angles is smaller than 180°, resulting in hyperbolic geometry. The circles are surrounded by an inverted image of this triangle. These two disjoint triangles are the target region $\bar{R}_T$ of the converged iterated mapping T that composes the inversions. Accordingly, the resulting image has two parts. The enclosed triangle generates a Poincaré disc representation of a tiled hyperbolic space (shown in blue). It is surrounded by its inverted image (shown in yellow). The border of the disc is the limit set of the circle inversions.

As you can see in these figures, the result of multiple inversions in three circles is equivalent to the dihedral group due to reflections at two straight lines in combination with the inversion in the remaining circle. The radius of the circle is irrelevant and thus three integer numbers (K, M, N) characterize these images. Moreover, shifting these numbers relates to rotations, $(K, M, N) \equiv (M, N, K) \equiv (N, K, M)$, and transpositions relate to mirror images, $(K, M, N) \equiv (K, N, M)$. The resulting tilings do thus not depend on the order of these numbers, although their appearance changes.

You find more details in my presentation at the Bridges 2018 conference [11, 12]. Do your own experiments using the "Kaleidoscope" browser app [13] or have a look at the Tiling Bot [7] on Twitter for more images.
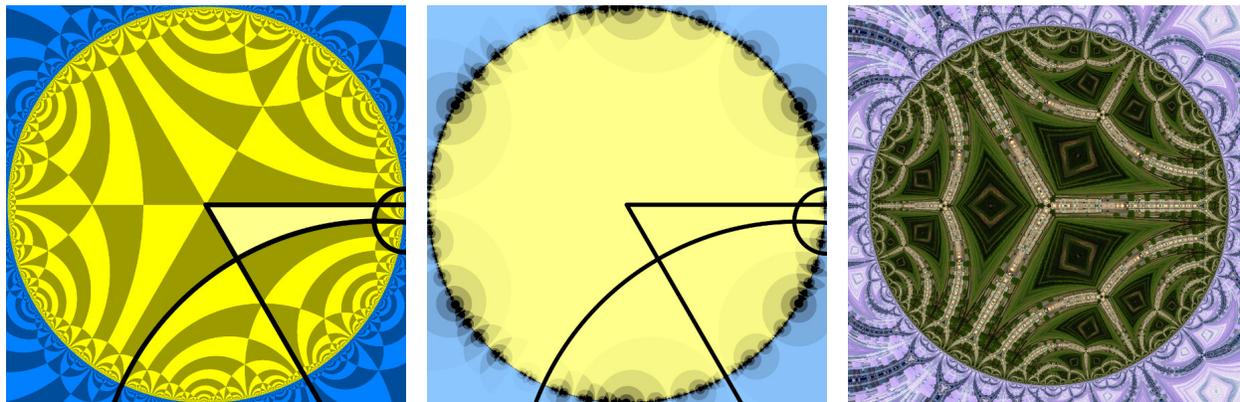
Figure 5: *Poincaré disc representation of a hyperbolic plane tiled by a parallelogram. The black lines show the generating elements. From left to right: The structure of the tiling, the limit set of the generating symmetries and a kaleidoscopic image.*

Four Circles as a Quadrilateral

Four intersecting circles can form a ring. Again we can use circle inversion to transform two of them into straight lines. Inversion at these circles is replaced by the dihedral symmetry generated by the lines. For
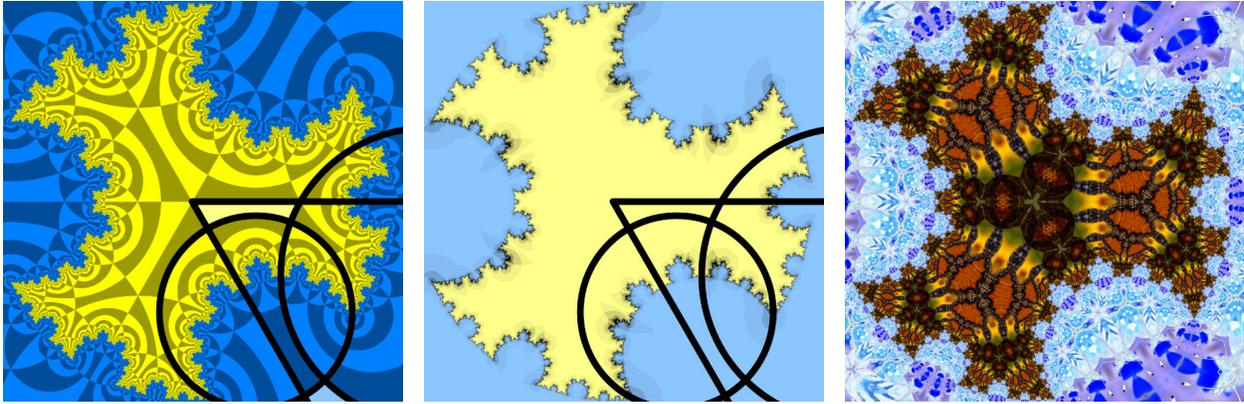
Figure 6: *Similar to Figure 5; but displacing the circles such that they are not straight lines in the same hyperbolic space. Thus the tiling becomes fractal.*

the other two circles the one-way maps invert only points from their inside to the outside. They connect the straight line as a chain. To determine the circles we have to choose values for the four intersection angles. If we want a hyperbolic tiling then we can use an arbitrary value for the radius P of the common Poincaré disc. This gives the condition $P^2 = d^2 - r^2$ for the distance d of a circle center from the intersection of the straight lines and its radius r. In addition, we can specify a value for the radius of one of the two discs. Elementary trigonometry then determines the positions and radii of the two circles. Figure 5 shows the tiling of the hyperbolic space inside the disc in yellow colors and its inversion outside in blue. The reflecting lines and inverting circles are shown as black lines. They enclose the basic hyperbolic parallelogram shown in pale yellow. It has two right angles and two angles of 60°. Its images are shown in yellow and dark yellow. These tilings are a generalization of the tiling of Euclidean space with rectangles. The center image of Figure 5 shows in dark color those points that require a lot of inversions to get mapped to these basic parallelograms. These points lie near the boundary of the Poincaré disc, which is the limit set of the reflections and inversions as discussed in the preceding section. Finally, Figure 5 shows a kaleidoscopic image resulting from matching an input image to the basic parallelograms. Here we more easily recognize the symmetries of the tiling.

We get more interesting results if we do not require that the circles fit a common Poincaré disc. We begin with the solution of hyperbolic geometry discussed above and move one of the circles towards the center. With increasing displacement, the deviation from hyperbolic geometry becomes stronger. Figure 6 shows such a result. Now, the quadrilateral at the center and its images (yellow and dark yellow) tile a fractal shape. Its outside is tiled by another quadrilateral and its images (blue and dark blue). Again, the center image of Figure 6 shows those points that require a lot of inversions in the circles to go to the basic quadrilaterals. Here we easily recognize the fractal nature of the boundary between the yellow and blue regions, which is the limit set of the circle inversions. For smaller deviations from a Poincaré disc it somehow resembles a Koch snowflake. Such limit sets have been discussed in detail before [1, 2, 4, 5]. You can do your own explorations using the "Kaleidoscope with quadrilaterals" browser app [14]. V. Bulatov shows similar fractal shapes in his fascinating presentation "Making Fractal Pendant in ShapeJS" [3].

Four Circles as a Circle in a Triangle

Having four circles we can use three of them to form a primary triangle of elliptic, hyperbolic or Euclidean geometry as discussed before. We put the fourth circle in the center of this triangle, such that it intersects or touches its sides. Its three intersection angles uniquely determine its radius and center. The fourth circle only inverts points from its inside to the outside. Adding this one-way map does not destroy the symmetric
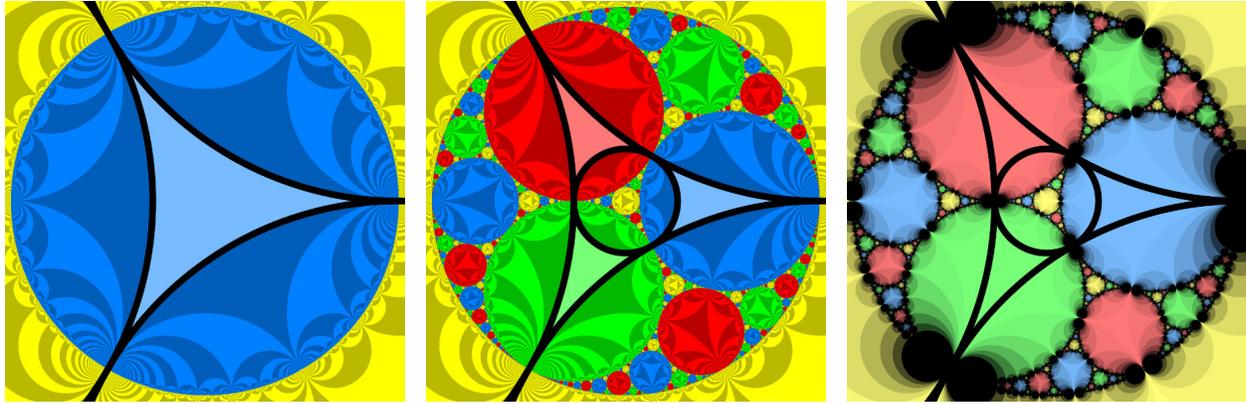
Figure 7: *Apollonius gasket as a fractal tiling of the hyperbolic plane generated by inversion in circles (black lines). From left to right: Structure of a tiled hyperbolic space resulting from three circles, structure of the gasket generated by four circles and the limit set of the inversions in the circles.*

structure generated by the primary triangle. Instead, we get three secondary triangles, one at each corner of the primary triangle. This creates more structure.

A well-known example is the fractal Apollonian gasket [6], which arises as the limit set of inversion in four tangent circles [5]. We first look at inversion in three touching circles that form a primary triangle with vanishing angles. This creates a Poincaré disc representation of tiled hyperbolic space tiled, see the left image of Figure 7. It is surrounded by its inverse as in Figure 4. Adding a tangent circle at the center creates three secondary triangles. Multiple inversion in the three circles of such a triangle again creates a Poincaré disc representation of tiled hyperbolic space. Multiple inversion in all four circles then fills the entire disc with images of these discs, see the center image of Figure 7. The right image shows the points that require a lot of inversions in dark color. They correspond to the limit set of the inversions in the four circles, which is actually the boundary of all these discs. This shows that the Apollonian gasket is based on a tiled hyperbolic space. Each tile bears the same fractal decoration. This resembles the boundary image of hyperbolic honeycombs [8].

We get a large variety of images for finite intersection angles. They are characterized by six integers (K, n, M, k, N, m), where the triplet (K, M, N) determines the geometry of the primary triangle. The other three numbers give the angles of intersection with the fourth circle. The integer n corresponds to the intersection with the side between the corners of the primary triangle with numbers K and M. The integers k and m are similarly defined. The triplets characterizing the three secondary triangles are thus (n, M, k), (k, N, m) and (m, K, n). The four triplets determine the geometries of the four triangles which will be combined by the inversion in all four circles.

Several symmetries make that seemingly different results are actually equivalent. Trivially, rotational symmetry makes that $(K, n, M, k, N, m) \equiv (M, k, N, m, K, n) \equiv (N, m, K, n, M, k)$ and mirror symmetry gives $(K, n, M, k, N, m) \equiv (K, m, N, k, M, n)$ for a mirror line going through the corner characterized by K. More interesting, an inversion in a circle with its center at K exchanges the inner fourth circle with the circle of the primary triangle that is not going through this corner, and thus $(K, n, M, k, N, m) \equiv (K, M, n, k, m, N)$. This means that all tilings resulting from the same set of four triplets are equivalent. There is no difference between primary and secondary triangles.

If all four triangles are hyperbolic, then they simply generate four Poincaré disc representations of tiled hyperbolic space. The disc resulting from the primary triangle is then covered by images of the discs generated by the secondary triangles and the inverted disc generated by the primary triangle. For finite intersection angles the discs at the corners of the triangles do not touch each other and we get a more complicated structure
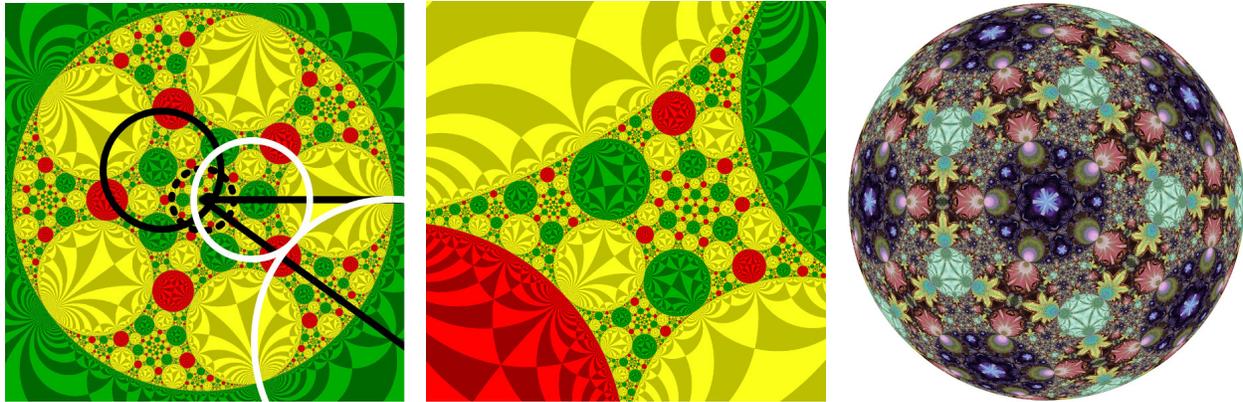
Figure 8: *Simultaneous fractal decoration of a Poincaré disc representation of a hyperbolic plane and a stereographic projection of a sphere with icosahedral symmetry. From left to right: Structure with generating elements, the gap between four discs and kaleidoscopic image of the sphere.*

than the Apollonian gasket. You can create your own examples using the "Kaleidoscope with a circle in a triangle" browser app [15]. R. Nelson and H. Segerman have published similar images using a different method [8].

Some of the four triangles can be of Euclidean or elliptic geometry. The example shown in Figure 8 results from $(5, 2, 3, \infty, 2, \infty)$ and the browser app [15] mentioned above. The primary triangle $(5, 3, 2)$ is made of the two straight lines and the large black circle in Figure 8. Itself alone generates a stereographic projection of a tiled sphere with icosahedral symmetry. The dotted line is the projection of the equator of the sphere. The fourth element is a small circle you can barely see. Its center lies on the upper straight line of the triangle for an intersection angle of 90° and it touches the other sides of the primary triangle. All three secondary triangles are hyperbolic and generate the red, green and yellow discs. The discs touch each other because of the vanishing intersection angles. Icosahedral symmetry makes copies of these discs that appear with different sizes because of the stereographic projection. The kaleidoscopic image at the right uses a normal projection of the sphere. It shows that the discs are actually all of the same size and have a dihedral symmetry at their center. The discs make a network of icosahedral symmetry. The green discs have five-fold rotational symmetry and lie at the corners of an icosahedron. The red discs with three-fold rotational symmetry lie at the centers of its triangle faces. The yellow discs with two-fold rotational symmetry lie at the middle of the sides of the triangles. They connect the green and red discs. Gaps arise between these discs. They are surrounded by two yellow discs, one red disc and a green disc, which may be inverted. All these gaps have equivalent decorations with icosahedral symmetry resembling the entire image, see the center image of Figure 8. The gaps in the decoration of these gaps are again filled by such decorations which results in self-similarity.

We note that an icosahedron having its corners at the north and south poles of a sphere is symmetric with respect to a mirror image at the equatorial plane together with a rotation of 180°. In the stereographic projection the mirror image at the equatorial plane is an inversion in the projection of the equator. Together with the D_5 symmetry at the center we obtain a transformation from $(5, 2, 3, \infty, 2, \infty)$ into $(5, 2, \infty, \infty, 2, 3)$, which corresponds to two lines and two circles that generate the same image. Figure 8 shows the straight lines in black and the circles in white. The primary triangle with $(5, \infty, 2)$ is now hyperbolic and on itself would generate a Poincaré disc representation of hyperbolic space. The elliptic secondary triangle with $(3, 5, 2)$ makes the icosahedral symmetry. Figure 8 thus is at the same time a fractal decoration of a sphere and of hyperbolic space. You can check this out using my browser app [15]. Even more exotic combinations are possible. In particular, you can create images that are fractal decorations of elliptic, Euclidean and hyperbolic

space. You can combine octahedral and icosahedral symmetry.

Summary and Conclusions

Images resulting from multiple inversion and reflection in intersecting circles and straight lines are examined. To simplify the search we are using inversion in an extra circle as an additional symmetry of the plane together with the usual translation, rotation, scaling and mirror symmetries. Two intersecting circles are then equivalent to two straight lines and only their intersection angle is relevant. For three circles and lines one simply gets the already well-known kaleidoscopic images. Depending on the three intersection angles, they are periodic decorations of spherical, Euclidean or hyperbolic space. Four circles and lines can form a quadrilateral or a triangle with a circle around its center. For a ring-like configuration surrounding a quadrilateral, we get a region with a fractal boundary covered by images of the quadrilateral. If three circles surround a hyperbolic triangle and the fourth circle lies on it, we get a large disc representing hyperbolic space with a periodic fractal covering by discs similar to the Apollonian gasket. These smaller discs represent hyperbolic space tiled by triangles. It is possible that such images have in addition elliptic or Euclidean geometry. I am discussing a self-similar decoration of a sphere with icosahedral symmetry, which is at the same time a fractal decoration of a tiled hyperbolic plane.

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