Dancing Euclidean Proofs: Experiments and Observations in Embodied Mathematics Learning and Choreography

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Abstract

Euclid's Elements (c. 300 BCE) has long been held to be an example of minimalist mathematical beauty. As mathematics educators interested in embodied learning, we want to share that beauty with our students (and mainstream dance audiences), to promote understanding and appreciation of Euclidean proofs in visceral, movement-oriented ways. This paper reports on a successful experiment in dancing Euclidean proofs in the context of a university mathematics education class, and explores philosophical, mathematical, and pedagogical dimensions of the process of dancing geometric proofs on the beach.

Introduction

In recent years, there has been increasing interest in embodying mathematical concepts and relationships through dance and movement, both as a way of creating intriguing choreography, and to help learners gain deeper understanding of aspects of mathematics [5,9,11,12,13,16]. Where once mathematics was considered a purely abstract and mental discipline, there is now growing cultural acceptance of mathematics as a pursuit that involves body *and* mind, experience *and* thought. Within the field of mathematics education, there is much new work on embodied ways of learning, some of which involves movement and dance [3,4].

Dancing Euclidean Proofs began as part of a 2018 University of British Columbia course on Mathematics History for Teachers. The course instructor (Gerofsky) offered a challenge to students to demonstrate the first proposition in Book 1 of Euclid's Elements as embodied movement, giving attention to the pedagogical affordances of teaching and learning geometry through movement and dance. students in the class (Duque and Milner) took up the challenge, and extended the work to creating and filming the first three propositions in the first book of Euclid on an ocean beach on a chilly winter's day. In undertaking this, the authors of this paper found that the process of embodying Euclidean proofs through dance raised many interesting ideas about the nature of geometric diagrams and perspective, of enacting geometric logic over time, and of humanizing mathematics (building on theoretical work from [10,15]), as well as questions of land-based learning and decolonization (in the context of Canada's Truth and Reconciliation process).

Note that, although there are several dance companies who have used Euclid's Elements as inspiration for abstract modern dance choreography [6,7], we believe our collaborative work is a unique example of mathematician/dancers embodying Euclidean geometric proofs and constructions in a precise way through movement, with pedagogical goals in mind.

Our Goals and Interests in Dancing Euclidean Proofs

The three co-authors are all involved in mathematics education and the performing arts, and integration of these two fields as a way to deepen students' math learning. Milner is a new secondary mathematics teacher and an accomplished violinist; Duque is an undergraduate student working with popular theatre and mathematics in her education work with children from refugee and other marginalized populations; Gerofsky is a university professor in mathematics education researching embodied, movement-oriented, and arts-based mathematics learning.

We share two related goals in this project:

(1) to create intriguing and beautiful artistic choreography based directly on Euclid, with the potential to make the beauty of Euclid's proofs accessible to mainstream audiences and students of mathematics through the physical beauty of dance; and

(2) to explore an embodied approach to mathematics learning that may help students understand and appreciate the beauty of Euclid's proofs in new multisensory, experiential ways.

The abstract, logical beauty of proof in the first four (geometric) books of Euclid's Elements has been recognized for millennia by those with the mathematical background to appreciate it. Artistic approaches, including Byrne's 1847 colour illustrated version [2], highlight the graphically visual beauty of the Elements, and make the intellectual beauty of the proofs more widely accessible in aesthetically pleasing form. We aim to bring a third realm of beauty to Euclid's Elements: the embodied, performative, temporal beauty of dance. For mathematics learners, exploration of multiple representations and modes of cognition has been shown to enhance and deepen mathematical understanding, building more holistic mathematical understanding through a knowledge of the equivalence of varied modes of representation [8]. We propose a pedagogical approach where math learners at secondary/post-secondary levels experiment with their own forms of dancing proofs of Euclidean and other geometric theorems, as a way of experiencing the logic of proof in more visceral ways. For mainstream audiences, cross-disciplinary connections across mathematics and dance may enable a more expansive sense of beauty and spark an appreciation of the beauty of mathematics. It would be exciting to see a dance company compose choreography presenting a fuller version of Euclid's Elements as an artistic performance piece.

Designing the Dances

Why Do Euclid's Proofs Lend Themselves to Dance?

Our first step was to make explicit for ourselves the rationale for translating these proofs as dance. We identified that:

• the step-by-step actions involved in geometric constructions, proofs and diagrams resemble a choreography in their sequential, active nature;
• the body's natural symmetries can be used to make (or draw) simple geometric shapes such as circles and triangles;
• proofs unfold over time in a way similar to the sequential moves of a dance;
• Euclid's proofs (and geometric proofs in general) are visual in nature, and these visualizations can be embodied through artistic representation in one way or another (as drawings, sculptures, paintings, textile arts — and as artistic performances, including dance).

Description of the Dances

Each dance represented one of the first three proofs from Euclid's Elements (Figures 1, 3, and 4 below). A short video of the three dances referred to here is available at <https://vimeo.com/330107264> and may help readers visualize the dances described here.

Dance 1/Euclid's Book 1, Proposition 1

On a given finite straight line, to describe an equilateral triangle.

The first proof shows that it is possible to construct an equilateral triangle with compass and straightedge. It begins with a given line segment which will form the base of the triangle. The given line is then used as a radius to describe two congruent circles whose centre points lie on each end of the line segment. In the dance, we walk toward each other and begin the proof with the given line, which, since it also serves as two radii, is represented by an outstretched arm of each dancer, one laid on top of the other. Although a tape measure may find difference, throughout this and the next dance we imagine that our arms are the same length. Throughout this dance we mirror each other exactly; thus, the given line is formed from one performer's left and the other's right arm. The vertical axes of our bodies are the two circle centres from which the overlaid radii extend.

Then we take a liberty: we extend our other arms behind us so that the radii become diameters — not found in Euclid's diagrams. Both dancers then spin a full 360 degrees in opposite directions, drawing circles in the air with fingertips. When the given line (created by our mirrored arms) comes to rest again at its origin, the trailing opposite arms keep turning until they meet at the point where the imaginary circles intersect. Thus, each dancer's second arm starts 180 degrees from the first through the vertex of the body and eventually closes in to 60 degrees. Though Euclid's original construction does not have the 'extra' diameter lines, at this point in the proof these arms take their rightful final place as two new radii, the two remaining sides of the desired equilateral triangle. Since all arms are equal lengths, we may trust that this triangle is indeed equilateral.

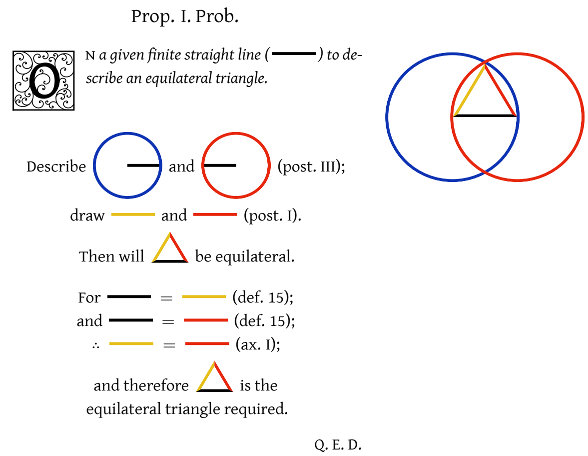

Figure 1: *Euclid Book 1, Proposition 1 from [2]*

Note here that the logic is not exactly parallel to the definitional and axiomatic propositions in Euclid's proof. Instead, we rely on our intuition that usually a person's two arms are equal in length, and we rely on our imagination that the dancers' arms are also equal. While we attempt to follow Euclid's steps and certainly a roughly logical progression of proof, yet what we know about our bodies also informs our choreographic choices.

Finally, since one revolution of the first dance takes just six seconds, it works well to repeat it as a sort of meditation. When the dance finishes, we drop our arms — the final equilateral triangle — and start again at the beginning by raising the arms which form the given line.

Dance 2/Euclid's Book 1, Proposition 2
From a given point to draw a straight line equal to a given straight line.

An element of the beauty of Euclid's Elements is the way that each proof builds upon previous proofs and constructions. The equilateral triangle from Dance 1 is used in Dance 2. Likewise, the line created at the end of Dance 2 is then used in Dance 3 to alter the original line.

The second proof shows that it is possible to draw a line segment equal to a given line segment from a given point. Due to the complexity of this proof we were inspired to include elements of the environment around us. The *given*s were represented by tangible materials found on the beach around us: the given line was a stick and the point was a rock. These are the first elements shown in the dance. In the following step, Carolina stands by the given point (the rock), and Sam stands by the edge of the given line (the stick); we both stretch our arms to create an equilateral

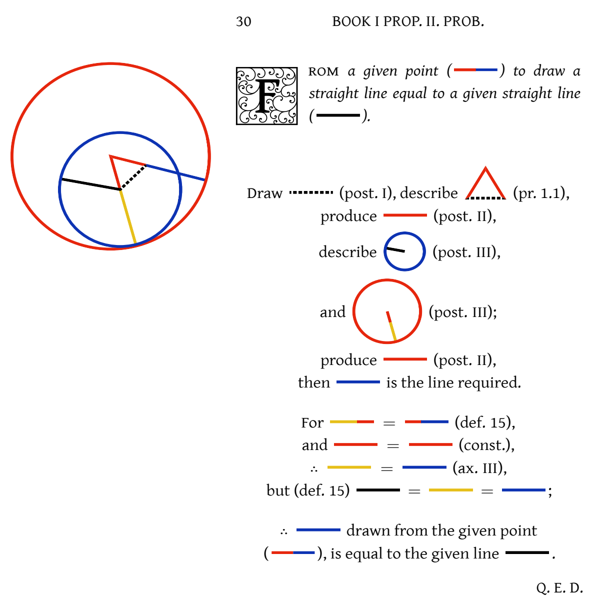

Figure 2: *Euclid Book 1, Proposition 2 from [2]*

triangle which is then carved in the sand with our hands. This construction is made possible by Proposition 1. Two shells are positioned at the edges of this triangle and will later become the centre points of circles.

The next step is to describe a circle using the given line as the radius and the triangle's vertex (the shell) as the centre. Sam uses his body as a compass by equating the distance between his legs to the length of the stick, and drawing the circle with one toe. The next circle has a radius created by the line between the other vertex of the triangle and the elongation of the remaining side of the triangle to reach the first circle drawn. For this second circle the compass is created by the cooperation of both dancers. By holding hands, Carolina's body is the axis from which Sam spins around to carve the second circle in the sand.

Figure 3: Still image, Dance 2: using the body as a compass *(from authors' video at https://vimeo.com/330107264)*

The final step involves proving that the line between the given point (the rock) and the second circle's circumference is the same length as the given line. Thus, the stick fits perfectly in that given space.

Dance 3/Euclid's Book 1, Proposition 3
From the greater of two given straight lines, to cut off a part equal to the less.

In Proposition 3, we are given two line segments of different lengths, and will show that is possible to cut off a part from the longer line segment equal to the shorter line segment. To achieve this, the first step is to draw a line equal to the shortest line from the edge of the longest line. We are able to do this based on the construction established in the previous proof.

In this dance, each line segment is represented by the extended arms of each dancer, with Sam's arms representing the longer line segment. Carolina dances from her original position to meet Sam's hand with her hand, and draws an imaginary circle using Sam's hand as the centre point and both her arms as the radius. The next step is to 'chop off' the remaining part of Sam's arm — that is, the given line.

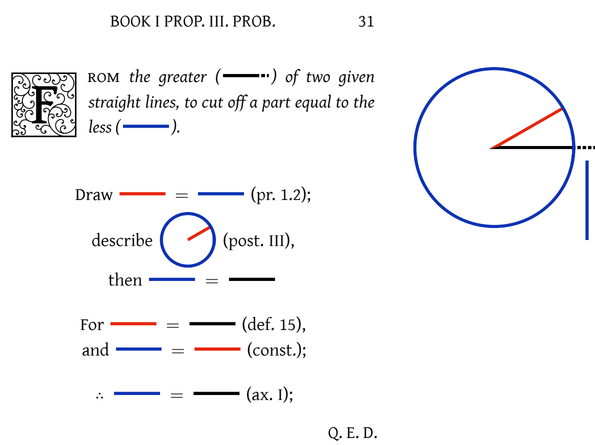

Figure 4: *Euclid Book 1, Proposition 3 from [2]*

Decision-making Processes in Creating the Dances, With Our Bodies and the Land

Any artistic representative endeavour, and any proof, involves decisions regarding clarity, practicality, and beauty, and our choreographic process demanded we make choices at forks in roads. We were constantly asking ourselves questions such as *What is most beautiful? What makes the proof most clear? What is practical?* Sometimes our bodies felt at odds with the geometric abstractions we were trying to represent, and it felt like trying to connect two south poles of magnets. Often, though, with some slight rethinking or repositioning, we found that the math and the dance actually fell together quite naturally, as — with a twist — magnetic north and south poles hold together on their own.

For instance, in Dance 1, we had originally spun each with one arm outstretched. After all, one usually constructs a circle with just one radius extended from a central point. However, using only one arm felt imbalanced and inartistic, choreographically, so we decided to try two. Not only did spinning with both arms out feel more dancerly, but it also ended up clarifying the construction, as the 'extra' arms met to complete the equilateral triangle. Moreover, since dancing this construction of Euclid's Proposition 1 takes all of six seconds, and the dancers end where they start, we thought that repetition could work well in performance. Thus a practical concern influenced an artistic concern, and furthermore, the repetition of this proof mimics the cognitive process of understanding Euclid's original diagram, the eyes and the mind going round and round the circles and steps of the proof.

The second dance, the longest and most complicated of the three, involved the most decisions and greatest challenges. We wanted to respect the integrity of this proof's structure while maintaining the fluid sense of the first dance. Asymmetrical, non-obvious, and requiring a careful sequence of logical steps, this proof felt to us more clinical than the first and thus more difficult to fit to a dance. While the first proof was simple and short enough to hold in the mind without leaving some lasting trace drawn in the air or on the ground, the second proof had more lines than we had limbs and involved more dependent parts.

As we played around with representations of circles, points, and lines using various combinations of our arms, legs, and bodies in a crowded hallway, we realized that we needed some way to record our movements. "Aha! Let's draw in the sand." Thus mathematical necessity and the constraints of our bodies opened a totally new dimension of our project: the land. Although this new element added beauty, it also added more possibility and thus more decisions. Where is the proof situated — at the level of our outstretched arms, on the ground, or both? Ought the lines be our arms, lines pre-drawn in the sand, or sticks? Are our bodies the points, or should we use rocks and shells? We came to some combination of these, and a poetic one at that. The givens in the proof are represented by a stick and a large rock; the other lines and circles are drawn in the sand and any new points needed are shells. The process — all the intermediaries in the construction of the desired line — would eventually be erased by tide and wave. The given point and the final line are preserved. Thus our final representation is both poetically and mathematically coherent, and yet required some wrestling to arrive there.

Theoretical and Experiential Observations

Perspectives, Dimensions, and 'Being the Proof'

One interesting experiential observation has to do with perspective. As we review Euclid's original constructions, we gaze from above at static images on the page. However, as we dance the proofs, and as a live audience might view them, we somehow enter the page. We imagine ourselves to be like the two-dimensional characters in Abbott's Flatland [1]. While dancing the proofs adds a *temporal* dimension to Euclid's original representation, the positionality of the dancers and audience (in the same plane) involves some loss of the third *spatial* dimension. In the filmed version of these dances, the use of a drone camera restores the high angle bird's eye view that replicates the point of view given in geometric diagrams; it would take some ingenuity, perhaps including ladders or viewing platforms, to restore this spatial perspective in a purely live dance performance.

Loss of perspective, though, is not always negative. Black and white photography clarifies what a colour picture might obscure; musicians sometimes close their eyes while playing to focus on sound. Limitation, whether chosen or accidental, can 'cut the noise.' Perhaps we see in this dimensional barter, then, that representation is always a tradeoff. As mathematicians, artists, and educators we constantly make decisions regarding what and how to emphasize; multiple and creative representations such as proof through movement add new tools to the box, new colours to the visible spectrum.

Another learning involving perspective has to do with the dancers as co-responsible for representing the proof as opposed to a single, differently-responsible student reading the Euclid text. If you sit down to study the Elements from a book, you are in a sense completely detached from its representation on the page. It is all there already, words and pictures, on the page, with or without you. However, in another sense, you are completely responsible for your own understanding of the proof. You must somehow, by will and mental effort, internalize the truth of it, recreating it in the mind, while remaining unattached to its representation. This, for us the typical experience of making sense of mathematical proofs, is quite different from what we felt as we choreographed and danced. As we danced, we were active agents responsible for the making and understanding the representation. The proof would not substantiate without us. We *were* the proof, in our collaborative dance, yet neither one of us was it by ourselves. Sam carried some of the proof in his movements, Carolina some more, the sand even more, and all of us together made it cohere. The shared nature of our danced construction, in both the weight and delegation of representative responsibility, contrasts the nonparticipation and sole cognitive responsibility of an individual learner of geometry.

Our understandings, too, differed from those of a lone student. The process of choreographing the dance proofs — making decisions, practicing, memorizing — both transformed and guaranteed the internalization and mental recreation of the proof. We needed to know in which order steps occurred, which were the necessary and dependent components, and in what ways. We needed to see how it all fit together in order to represent it in a new way. Moreover, as we tried and discussed different representations, as we danced in the hallways of the university and then on the beach, and even as we wrote this paper, we became more and more familiar with its structure. We suppose that this is a result not just of embodying proofs in a new and creative way, but also of the social and negotiated nature of our process.

Metaphor, Representation, and Imagination

Math done on paper is a representative process that relies on the mathematician's imagination to reconcile the limitations of representation. Euclid's proofs, for example, take points to be infinitely small and lines to be infinitely narrow and long. However, the points we draw on paper have real size, the lines real width and length. In the same way, the audience must reconcile the 'misrepresentations' of our dancing bodies. Our roughly cylindrical forms are clearly more spacious than infinitely small points, and even — from our point of view — wholly different from ink dots on a page. To a mite or a molecule, though, a dot on a page may as well be a baseball diamond; next to the infinitely small point, pencil dots and human bodies differ only as a matter of perspective. Does it matter, then, that in forming an equilateral triangle our torsos are not 'real' corners? No. The corners in Byrne's illustration are not 'real' corners, either. Dance math, like paper math, like any math, requires stunt doubles [10]. From the first and second to the third dance, the observer's attention must transition between contrasting imaginative representations; she must employ a flexibility of mind which to us feels much like that which one needs to abstract or to move quickly between representations.

Moreover, the observer's imagination is further engaged, particularly in the first and third dances, as she must hold in her mind the circles drawn in the air by our spinning arms. It is as if an imaginary residue were left by our trailing fingers. Surely line is always important in dance, and yet here the visualized contrails serve not just an aesthetic but also a cognitive purpose in the observer's apprehension. Math and art work together, buffering each other, in the minds of dancers and observers alike.

Part of our learning in embodying Euclid's proofs through dance, then, is that doing mathematics involves metaphors sustained by the agreement between imperfect embodied experience and the imagination of abstract representation. When we dance math (rather than writing or drawing it), the involvement of our whole bodies offers a stronger experiential component, and demands different capabilities from the imagination of the dancer-mathematician.

Temporality of Dance and Proof/Humanizing Mathematics through Dance

Pimm compares certain practices of mathematical proof (which attempt to eliminate evidence of human involvement) to historical styles of painting which value the invisibility of the painter. He claims that "In parallel with the hiding of brushstrokes suggesting a timeless and agent-less 'work of art', geometric construction lines are also to be made to disappear, once their task has been carried out" [10]. Although the erasability of our process in a high tide may appear to further the myth of mathematics' independence of humanity, yet our agency in re-enacting the proofs through dance with our very human bodies boldly declares context, performance, situation. The impermanence of drawings in the sand nods not to a disembodied Bourbakian mathematics but rather to the notion of construction itself, in its means and ends; moreover, our (particular) bodies assert human positionality as necessary in any math educational and/or performative endeavour.

We may view our project, then, as a disciplinary practice which reveals the 'human mathematical agent.' As we embody mathematical entities, the dance becomes symbolic of mathematics *as* humanity and humanity *as* mathematics. We are human agents not just *wrestling with* but also *becoming* the mathematics we do. In yet another sense, our performance is not only a performance in the usual choreographic sense, but also a *rendition* of Euclid's first three constructions. As a musical composition becomes available to the listener through performance, so Euclid's proofs become available as shared experience through dance.

Pimm also notes that images, which are "at least two-dimensional," are thus "not easily mapped onto a time sequence," as time is typically represented in one dimension [10]. The temporality of dance, in comparison to the 'all-at-once-ness' of image, carries both affordance and limitation. Due to representational tradeoff, we sacrifice in our dance a bigger picture as we are 'stuck' in moments. However, we gain through temporality an analogy to the cognitive process of understanding proof and thus strong ties also to pedagogy. Whether a proof is presented as a series of narrated and dependent steps (in which case temporality affirms itself yet again) or as a whole visual image or a combination of the two, a learner takes time to make connections, follow steps, and understand contingent parts. So dancing spreads Euclidean constructions across space *and* time, and as a written or drawn proof may symbolize the mathematicians' work, so dancing may draw attention to the math learners' work.

Dancing Euclid on the Land/Environment and Mathematics

Our decision-making process involved using the environment around us as part of the mathematical proof. A vital aspect of embodied knowledge is to acknowledge the space that shapes our movements. The body does not move in a vacuum, but in response to stimuli from the land and place. Likewise, the natural world offers countless possibilities to realise the mathematics of beauty: the skeleton of leafless trees, the spirals on shells, the rhythm of waves. Our initiative to include natural elements in Dance 2 came from the idea that the art in mathematics is found in our bodies as much as it is found in the natural world around us. It just takes a degree of ingenuity, passion and curiosity to unveil such hidden aesthetic patterns.

While extending our choreography to include and depend on natural environments afforded us new and helpful mathematical and aesthetic possibilities, we also encountered several unexpected limitations. As we were creating the dances in the sterile built environment of university hallways, the affordances of dancing on the beach struck us, and we formed idyllic pictures of what incorporation of 'the land' might feel and look like. For instance, we had originally hoped to draw circles in dry sand with water dripping from a vase in our hands as we spun. We liked the temporariness of water on sand, construction lines soon to evaporate in the sun; moreover, we wanted to keep the drawing at the height of our shoulders because we imagined stooping to be awkward, difficult, and clumsy, somehow interrupting the flow of the dance. However, although we filmed on the beach on a sunny day, it was Vancouver in December, and thus our insulated expectations met cold and damp reality. It was not raining, but the sand was dark and wet, forcing us to revert to other plans. Nearer the equator or July, the dance might have looked differently; but as it was, our particular geographical setting became an active limiting agent in our representation.

Other less mathematically related environmental phenomena also caught us off guard. Bare feet numbed quickly in the sand, seagulls found our food, and children flew kites loudly within earshot and within our camera's view. We began this project with a notion of our own control; we were choreographing, we were planning, we were understanding and representing Euclid — we were somehow in charge of it all. Yet when we took the dance outside of the temperature-controlled Faculty of Education, the locus of control shifted from us to somewhere outside of us, perhaps even to the dynamic relationship between us and nature. Here Simpson's 'land as pedagogy' [14] is especially relevant as we learned and danced on the unceded, traditional and ancestral territory of the Coast Salish Indigenous peoples. With the lens that the land is our teacher, we may perceive our choreographic, mathematical, and performative processes as small lessons in the decolonization of mathematics. Traditional Western notions of human control and domination — here over the act of mathematical representation and the process of mathematical understanding — are confronted and subverted by 49th parallel weather.

Simpson's reflections invite us to engage with knowledge differently. Part of our challenge was to embrace an epistemology where knowledge resides in the non-human phenomena that surround us. It involves welcoming a wider conversation between the land, the dancing body and the mathematical mind; to us, that is where art truly takes place.

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