

Embroidery of a Hyperbolic Fish Pattern

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Abstract

We have used embroidery to render a fish pattern in the Poincaré circle model of hyperbolic geometry. We will explain how we arrived at using the embroidery process for displaying such a pattern, describe that process, and then show the resulting fiber work.

Introduction

Our objective was to use embroidery to create a pattern in the Poincaré circle model of hyperbolic geometry. We chose an Escher-like fish pattern because it had features such as filled areas, long thick arcs, and interior detail that would be amenable to production by a computer-controlled embroidery machine. We succeeded in our quest as Figure 1 shows.



Figure 1: *An embroidered hyperbolic fish pattern.*

The inspirations for this work arose over a number of years. The first was the collection of four hyperbolic “Circle Limit” patterns of M. C. Escher. In the early 1980’s these patterns inspired Dunham and his students to write a computer program that could replicate those patterns. We will discuss this in the next section. Then later, there were two fiber artists working in hyperbolic geometry who were also inspirational. We will discuss them in the following section. Next we examine the embroidery process and show the final result. Finally we summarize and indicate possible directions of future work.

M.C. Escher Inspiration

In the late 1950’s the Dutch artist M. C. Escher created four circular woodblock patterns *Circle Limit I*, *Circle Limit II*, *Circle Limit III*, and *Circle Limit IV*. Although his intention was simply to create patterns in which the motifs became ever smaller toward a circular limit, they could also be interpreted as patterns in the Poincaré circle model of hyperbolic geometry. More than 100 years ago David Hilbert proved that there was no smooth embedding of the hyperbolic plane in ordinary Euclidean 3-space [4]. Thus we must rely on models of hyperbolic geometry, such as the Poincaré circle model. In this model, the points are interior points of a bounding circle and the hyperbolic lines are represented by circular arcs perpendicular to that bounding circle (including diameters). See Figure 2 in which some hyperbolic lines are superimposed on our rendition of Escher’s *Circle Limit I* pattern. Escher’s *Circle Limit III* and *Circle Limit IV* are the best known [3] whereas *Circle Limit II* (a pattern of crosses) is less well-known. Figure 3 shows our rendition of Escher’s *Circle Limit IV*; we also show our rendition of *Circle Limit III* below in Figure 4.

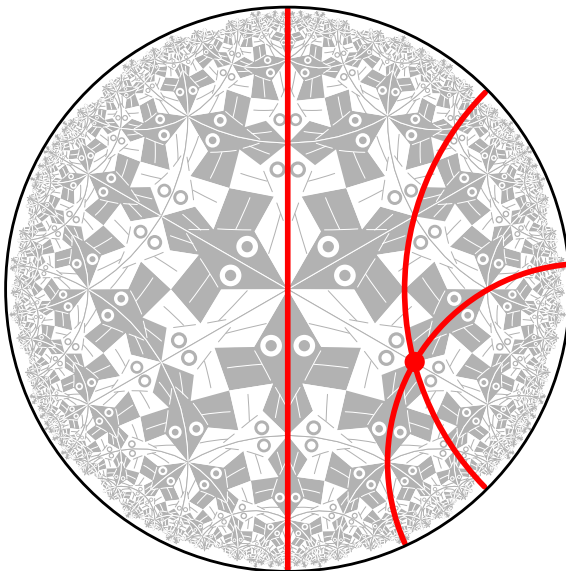


Figure 2: *The Poincaré circle model.*

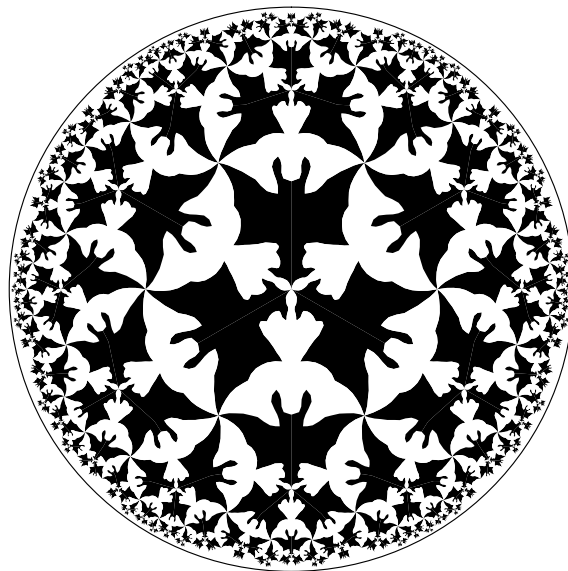


Figure 3: *Our Circle Limit IV rendition.*

For integers p and q greater than 2, there is a regular tessellation $\{p, q\}$ of the sphere, Euclidean plane, or hyperbolic plane by regular p -sided polygons meeting q at a vertex depending on whether $(p-2)(q-2)$ is less than, equal to, or greater than 4, respectively. Many of Escher’s patterns are based on regular tessellations. Both *Circle Limit I* and *Circle Limit IV* are based on the $\{6, 4\}$ tessellation; *Circle Limit II* and *Circle Limit III* are based on the $\{8, 3\}$ tessellation.

With this inspiration, our goal was to create a computer program that could reproduce Escher’s *Circle Limit* patterns. We succeeded in this goal [1] and more because the numbers p and q were parameters of the program, which allowed us to create new Escher-like patterns with different values of p and q . In particular

we created a new *Circle Limit III* like pattern, which we call *Five Fish*, based on the $\{10, 3\}$ tessellation, with five fish meeting at right fin tips instead of four fish as in *Circle Limit III*. Figure 4 shows our rendition of Escher's *Circle Limit III* next to our *Five Fish* pattern in Figure 5 for comparison.

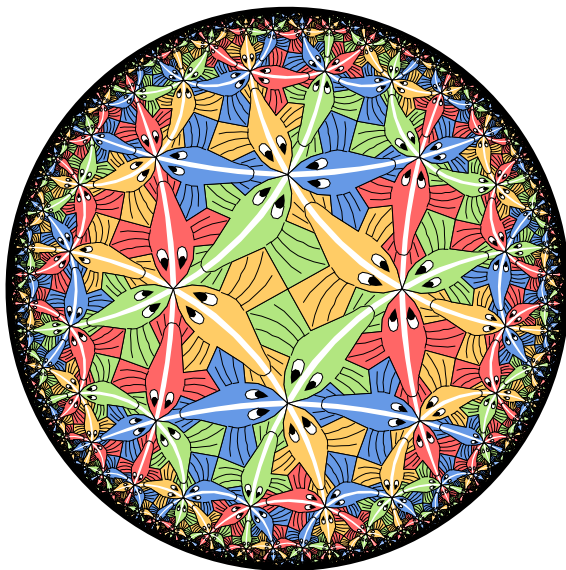


Figure 4: Our rendition of Circle Limit III.

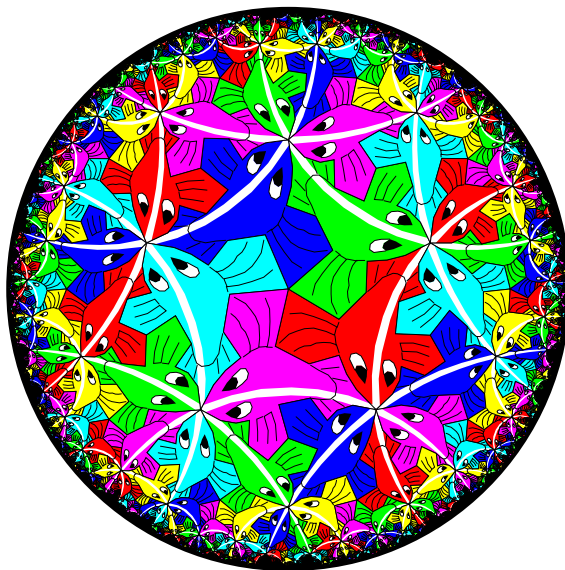


Figure 5: Our Five Fish pattern.

Inspiration from the Fiber Arts

Since the late 1980's Dunham had become aware of fiber artists who had created art works in 2-dimensional hyperbolic geometry. Dunham first interacted with the Australian hooked rug maker Tony Bomford in 1988. Bomford completed 17 hooked rugs, five of which were based on the Poincaré circle model — the rugs that he numbered 12, 13, 15, 16, and 17. He had completed *Rug 12* and *Rug 13*, both based on the $\{6, 4\}$ tessellation, when Dunham made contact. Dunham provided Bomford with printouts of the $\{5, 4\}$ tessellation, which he used to create *Rug 15*, and the $\{7, 3\}$ tessellation, which he used for *Rug 17* [2].

In the early 2000s' Dunham also collaborated with Mary Williams, a quilter who had produced a number of Euclidean quilts, but had not ventured into hyperbolic geometry before. As with Bomford, Dunham provided her with a printout of the $\{5, 4\}$ tessellation, from which she made the quilt *Poincaré* which was shown at the 2006 Joint Mathematics Meetings Art Exhibit (galleries of the Art Exhibits from the Joint Mathematics Meetings and the Bridges conferences are viewable at [5]).

Figures 6 and 7, below, show Bomford's *Rug 17* and Williams' *Poincaré* quilt respectively. In examining Bomford's rugs and Williams' quilt, it is interesting to compare the effectiveness of rug hooking versus quilting to portray patterns in the Poincaré circle model. Rug hooking is well-suited to creating the small motifs toward the edge of the bounding circle, however it is somewhat tedious when used to fill large areas in the center. On the other hand with quilting, large pieces of cloth are easy to deal with in the center of the bounding circle, but creating small motifs near the edge becomes challenging.

Finally, Dunham visited an embroidery shop in Varanasi, India in March of 2018 and wondered if embroidery could be used to render a hyperbolic pattern because areas could be filled and small details could be created with single stitches. Then in September of 2018, he got in touch with Shier who agreed to the proposed project.

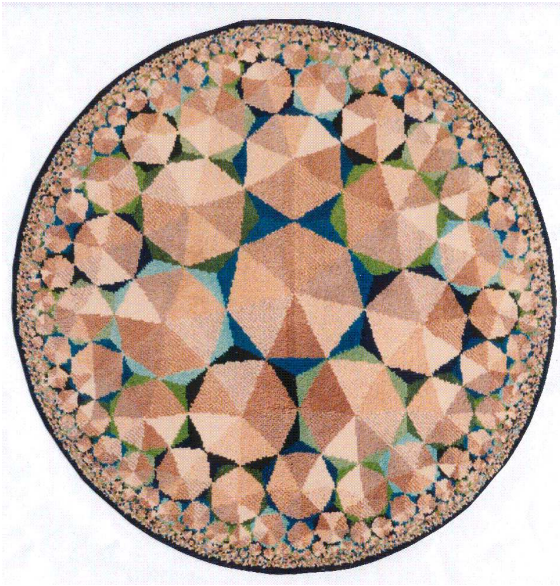


Figure 6: *Tony Bomford's Rug 17.*



Figure 7: *Mary William's quilt Poincaré.*

Creating the Embroidered Hyperbolic Fish Pattern

An automated embroidery machine can make three kinds of stitches: run stitches, satin stitches, and fill stitches. Run stitches are done with a single width of thread and implement polylines, such as the spines on the fish fins. Satin stitches form thick linear features, zigzagging back and forth across the linear “axis”. Satin stitches are ideally suited to stitching the thick backbone lines. In fact those backbone lines required variable width satin stitches near the center, constant width satin stitches half way to the boundary, and run stitches near the edge. The fish pattern is the only pattern Dunham and his students created that has those thick linear features, hence was one of the reasons we chose that pattern to implement. Fill stitches do what they say, they fill polygonal areas such as the fish bodies. But those fill stitches need to be oriented for each polygon (fish). This was done by consistently aligning those stitches with a fish feature, one of the fin edges. Using a constant orientation over the entire pattern produced aesthetically unsatisfactory results, and using varying orientations limited the distortion from stretching. Of course the filled fishes had to be stitched first and then the details such as the fin spines and backbone arcs were stitched on top.

The actual design and implementation of the final pattern took place over three successively larger versions, starting with a four inch diameter “proof of concept” version, and ending with the final 14.5 inch diameter version. Each version required a two-step process: the design of the pattern using Floriani Total Control design software that creates the input file (encodings of the different kinds of stitches) that drives the automated embroidery machine, and the actual stitching out of the pattern, done by a Husqvarna Epic embroidery machine. For each version Dunham sent Shier a jpg file of the fish pattern at the given size and at progressively higher resolution.

As mentioned, the first step is to design the pattern. The Floriani Total Control software can take a jpg file as input, but that is used only for tracing the stitches which must be done at as high a resolution as possible because the pixels on the jpg pattern don't usually match the coordinates of the embroidery machine, which are accurate to approximately 0.1 millimeters. In order to encode some of the smallest features for the largest pattern, the jpg file had to be magnified by as much as 800 fold. The design process took Shier two weeks for the 10.5 inch diameter pattern. To make the 14.5 inch pattern, an additional two weeks were required to refine that description by more accurately specifying coordinates and by adding smaller fishes. Figure 8

shows the design pattern of Figure 1 displayed in a screen capture of the Floriani Total Control software.

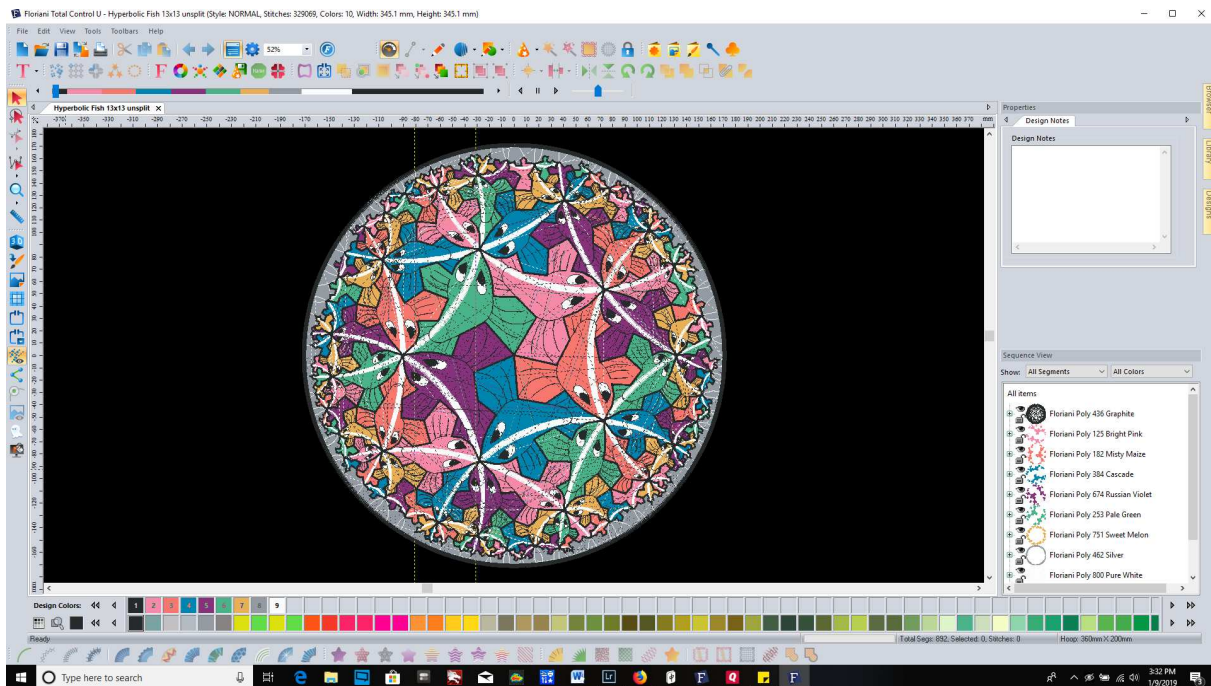


Figure 8: *Floriani Total Control embroidery software.*

The stitch encodings are saved in a file to be used as an input file to the embroidery machine. There is another detail to deal with in the design process: how to handle the area between the smallest fish and the bounding circle. Escher solved this problem by making small nicks in his wood blocks with dentist tools; Shier used a solid gray color, the average of the fish colors, for that area (actually that area was broken up into 10 fill areas). There are also a few design considerations that look forward to the actual stitching step, which we will discuss next. One such design feature is noting that the fish along a backbone line are all the same color and then filling the fish from tail to head, so that the next fish can be started where the current one finished — this saves changing the thread and repositioning the stitch point. This complicated design process is the most time-consuming of the two implementation steps.

The second step in creating the pattern is to actually stitch it out using an automated embroidery machine, a top-of-the-line Husqvarna Epic embroidery machine in our case. For our first “production” version of the pattern, Shier used a 260mm by 360mm hoop, the largest single position hoop the Husqvarna Epic supports. This allowed for the creation of a 10.5 inch diameter pattern. Since there are areas covered with more than one level of stitching, she needed a heavy fabric for backing and chose cotton duck canvas. But cotton stretches, so to counteract that, Shier used ironed-on Floriani Heat n Sta stabilizer and used a second layer of regular tear-away stabilizer underneath. For added stabilization, some of the lines were stitched before any of the stitches that show. Figure 9 shows the Husqvarna starting to stitch one of the red fish, and Figure 10 shows the finished 10.5 inch pattern. Notice that it lacks some of the details of the larger 14.5 inch pattern shown in Figure 1. Dunham told Shier that the colors in Figure 5 were only for guidance; she was free to choose her own colors. Unlike Bomford who ordered his yarn with special dyes, Shier was satisfied with the spectrum of commercially available thread. She used colored thread slightly variegated with white, which produced much more pleasing results than constant color thread.

To create the pattern of Figure 1, Shier used a 350mm by 360mm Husqvarna Majestic Hoop, the largest one available. That hoop has the advantage that it can create a bigger pattern than the 260mm by 360mm



Figure 9: The Husqvarna starting a red fish.



Figure 10: The finished 10.5 inch pattern.

hoop, but it has the disadvantage that it can only work on half of the pattern at once. After some work on one half was finished, the hoop has to be rotated 180 degrees and reattached to the embroidery machine to work on the other half. After designing the 14.5 inch pattern, that pattern had to be split into its two parts: the left and right semicircles. Then the two halves needed to be matched. First this was done just for the bounding circle, then it was done for the outlines of the fish. Finally the design was ready to be stitched, but to add stability the stitching was shifted between the left and right semicircles a number of times. This lengthened the stitching process, and in addition to changing thread (and watching for thread breaks), the stitching step took the better part of three days to complete.

Summary and Future Work

We have shown an embroidered fish pattern in the Poincaré circle model of hyperbolic geometry. There are certainly other other hyperbolic patterns that could be rendered in embroidery. Another possibility is to embroider Escher-like patterns on triply periodic polyhedra made of stiff cloth. We would like to explore these possibilities.

References

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- [2] D. Dunham. "H. S. M. Coxeter and Tony Bomford's Colored Hyperbolic Rugs." *Bridges Conference Proceedings*, Banff, Alberta, Canada. July 31 – Aug 3, 2005, pp. 497-504, <http://archive.bridgesmathart.org/2005/bridges2005-497.html>.
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