Tuti-Like Interweaving

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Abstract

Tuti Interweaving is a systematic approach for weaving halftoned (monochrome) images in Jacquard looms, presented by Ahmed and Deussen in Bridges 2017. It uses alternating black and white yarns for both warp and weft, and applies a preset weaving structure for half of the fabric, leaving the rest to render the desired image. In this paper we discuss alternatives that use white weft and black warp, or vise versa. We also address a situation where Tuti Interweaving would not guarantee a sturdy fabric as supposed.

Introduction

Weaving design is a research area that lends itself well to mathematical study. The problem statement is simple: we want to create aesthetic weaving patterns under the physical and practical constraints of the looms and the weaving process. This problem has attracted many scholars over the time, including Grünbaum and Shephard [12], authors of the famous "Tilings and Patterns" textbook [13], Andrew Glassner [10], a renowned figure in computer graphics, and Ralph E. Griswold [11], the late computer scientist. The Bridges conference published many articles in this area [1, 2, 3, 4, 5, 8]. For the interested reader, sarah-marie belcastro maintains an excellent compilation of “mathematical articles on fiber arts” [6].

In a previous Bridges paper [4] we introduced *Tuti Interweaving*, a systematic approach for weaving halftoned (monochrome) images in Jacquard looms, using alternating black and white yarns for both warp and weft, and a fixed weaving structure for the 50% of the fabric where same-color yarns intersect, leaving the rest to render the desired image. Alternating the yarn colors is a costly operation. Furthermore, the proposed idea suffers from a subtle flaw that leads to loose fabric in large black or white regions of the image.

In this paper we show that alternating the black and white yarns is actually not needed to use the principle that underlies Tuti Interweaving. We enumerate eight weaving structures that serve the job with white-only warp and black-only weft, or vise-versa. We then discuss which of these structures guarantee a sturdy fabric and which may fail with certain images. We start with a brief discussion about the weaving process and its emerging constraints. Readers who are familiar with our previous Bridges papers may skip the following section.

The Weaving Process

A simple woven fabric is a uniform mesh of yarns that run along the fabric, called the “warp,” and yarns that run across, called the “weft.” At each intersection, either the warp or the weft yarn runs above, and we see the color of that yarn, while the color of the yarn running beneath is seen on the other side. The whole piece of fabric can be abstracted as a two dimensional matrix with binary entries: each row corresponds to a weft yarn, each column corresponds to a warp yarn, and each entry indicates whether the warp yarn is (0) above or (1) beneath. For example, the weaving matrices for two common fabrics are:

$$\mathbf{S}_{\text{plain}} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & \cdots \\ 1 & 0 & 1 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & 1 & 0 & 1 & \cdots \\ 1 & 0 & 1 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & 1 & 0 & 1 & \cdots \\ 1 & 0 & 1 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \mathbf{S}_{\text{twill}} = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 1 & \cdots \\ 1 & 0 & 0 & 1 & 1 & 0 & \cdots \\ 0 & 0 & 1 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & 1 & \cdots \\ 1 & 1 & 0 & 0 & 1 & 1 & \cdots \\ 1 & 0 & 0 & 1 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The weaving matrix $\mathbf{S}$ reflects the structure of the woven fabric, but not necessarily what is actually seen by the eye. Colors and other properties of individual yarns are superimposed on the weaving structure to determine the final look of the fabric, which could also be represented as a matrix $\mathbf{F}$. Incorporating actual color information is a simple look-up process. A 0 in the weaving matrix is replaced by the color of the corresponding warp yarn, and a 1 is replaced by the color of the corresponding weft. Algebraically:

$$\mathbf{F}[i,j] = \mathbf{Clr}_{\text{weft}}[i].\mathbf{S}[i,j] + \mathbf{Clr}_{\text{warp}}[j].\bar{\mathbf{S}}[i,j]\,, \tag{1}$$

where the over-bar means taking the binary complement; that is, replace a 0 by a 1 and a 1 by a 0.

Woven fabrics are produced by a device called a loom. It weaves the fabric row by row, and for each row it raises the warp yarns designated to stay above, according to the weaving matrix, and passes the weft yarn beneath them. Weaving an arbitrary matrix requires full control over each individual warp yarn. This capability is provided by "Jacquard" looms, which are very expensive. The alternative is to raise warp yarns in groups rather than individually. A "dobby" loom uses "shafts" to control groups of warp yarns. Designing for dobby looms was discussed in our earlier Bridges papers [1, 2, 3], then we moved to Jacquard looms in [4, 5], as well as this paper. We confine our discussion to a single layer of fabric and a 'basic' Jacquard loom — one that offers individual lifting control over warp yarns, no more.

Tuti Interweaving

At first glance, the yarn-by-yarn control in Jacquard looms may sound like we are free to weave an arbitrary pattern, or an image. There are, however, still some important constraints that limit this ability. First of all, the color of the yarn remains fixed along the whole row or column, which drastically limits the choices of weaving design in comparison to tiling design: from a 2-dimensional matrix with arbitrary entries, to a pair of 1-dimensional vectors convolved with a binary matrix, as described in Eq (1). Secondly, we have the "floats" problem, which relates to the fabric itself rather than the weaving mechanism. In weaving terminology, a float is a loose strand of yarn. It results in a weak fabric, and the strand of yarn becomes prone to tearing and snagging. Floats are reflected as long runs of 0s or 1s in the weaving matrix; whether in rows (weft), columns (warp), or both.

Tuti Interweaving is a systematic approach to overcome these problems [4]. If we use alternating black and white yarns in both the warp and the weft, we can take advantage of the degree of freedom availed by having both colors coming along both axes. We use a fixed weaving structure for the white-white and black-black intersections, devoting them to solve the floats problem. Once the floats problem is excluded, we may now exercise our freedom in the remaining 50% intersections to control the color in accordance with the image we would like to render. The core concept is simple and straightforward: we plain-weave the black yarns together, plain-weave the white yarns together, and interweave the two colors in accordance with the underlying image; see Figure 1. Since each second yarn is the same color, the plain weave guarantees a

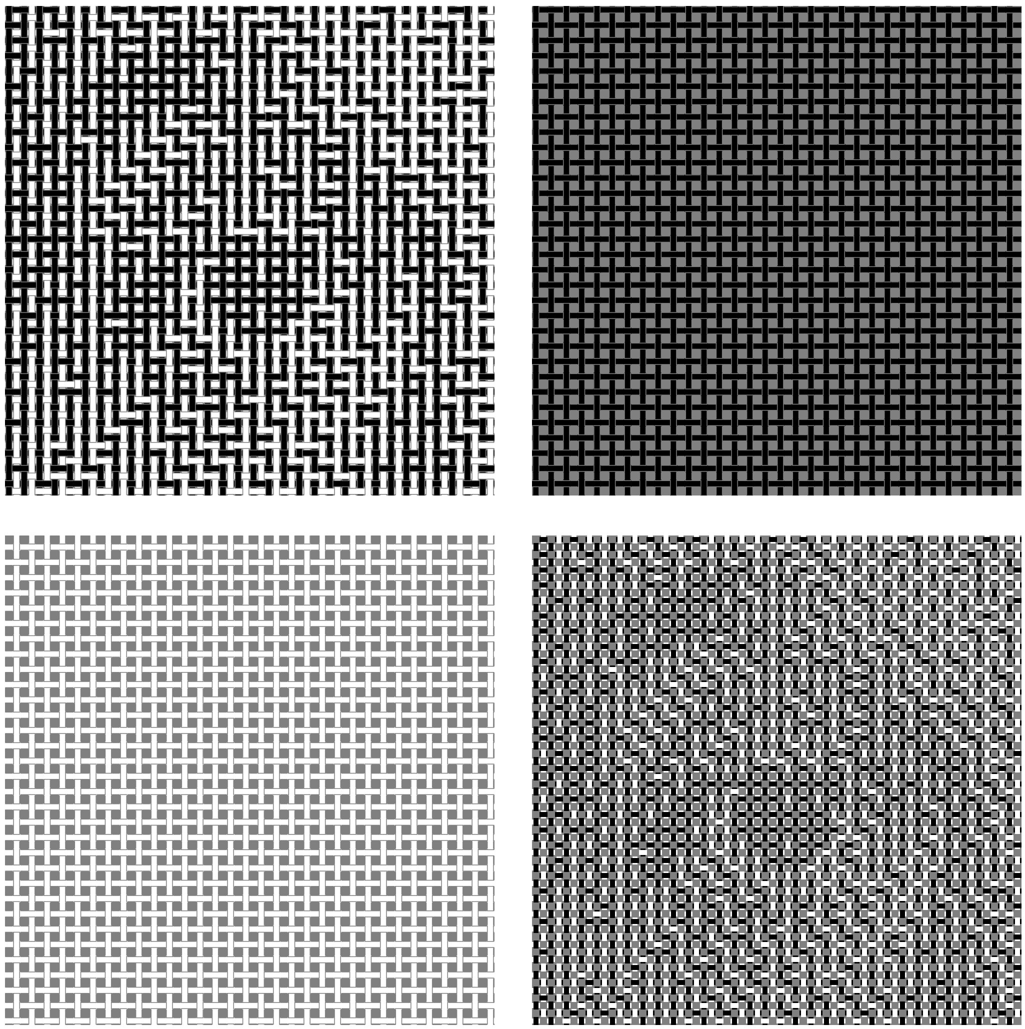


Figure 1: *(top-left) A Tuti interwoven image, obtained by (top-right) plain-weaving the black yarns, (bottom-left) plain-weaving the white yarns, and (bottom-right) interweaving the two colors in accordance with the underlying image. The pattern in this illustration is extracted from the right eye area in Figure 2 (top-left).*

maximum float length of 3. The weaving structure can be summarized in this 4×4 weaving block:

$$\mathbf{S}_{\text{Tuti Interweaving}} = \begin{pmatrix} ? & 1 & ? & 0 \\ 1 & ? & 0 & ? \\ ? & 0 & ? & 1 \\ 0 & ? & 1 & ? \end{pmatrix}, \tag{2}$$

where the question marks indicate the arbitrary positions that follow the halftoned colors of the image.

While the concept is simple and neat, Tuti Interweaving bears substantial operational costs for using alternating yarn colors in *threading* (feeding the warp) and *treadling* (feeding the weft). The idea, hence the name, came from an earlier idea, Tuti Weaving [3], that used a similar structure of alternating black and white yarns. In a second look, however, we can see that the alternating yarn colors were only needed to visualize the concept. It works as well to use all-white warp and all-black weft, and allocate a fixed weaving structure to maintain a float-free sturdy fabric, leaving the rest to render the desired image. In the following section we enumerate alternatives that also use 4×4 weaving blocks.

Tuti-Like Interweaving

Assuming an all-white warp and all-black weft, our target is to design float-free 4×4 weaving blocks using 50% of the entries, leaving the rest for the halftoned image. This may sound like a huge variety, but it actually reduces down to only eight distinct choices! We will outline this enumeration process to familiarize readers with non-mathematical background that there is a lot of common sense behind mathematics. "Without loss of generality" is a common expression in mathematical contexts, and we use it here to fix a 0 in the bottom-left of our weaving blocks. There is more than one way to justify this. For example, if a weaving block has a 1 at this place, then looking from behind toggles 0s and 1s, though we have a side-to-side reflection. Looking at a mirrored image then fixes this reflection, and it is still reasonable to think of this as the same weaving structure. Alternatively, we may tile our weaving blocks, and slice a new block to start at a 0, and we are still talking about the same fabric.

Thus, without loss of generality we may assume a 0 at the bottom-left, and start to populate the block with 0s and 1s to have one 0 and one 1 in each row and each column. If we consider horizontal, vertical, and diagonal reflections to be the same structure, we end up with the following eight distinct blocks:

$$\mathbf{S}_0 = \begin{pmatrix} ? & ? & 1 & 0 \\ ? & ? & 0 & 1 \\ 1 & 0 & ? & ? \\ 0 & 1 & ? & ? \end{pmatrix}, \quad \mathbf{S}_1 = \begin{pmatrix} 1 & ? & ? & 0 \\ ? & ? & 0 & 1 \\ ? & 0 & 1 & ? \\ 0 & 1 & ? & ? \end{pmatrix}$$

$$\mathbf{S}_2 = \begin{pmatrix} ? & 1 & ? & 0 \\ 1 & ? & 0 & ? \\ ? & 0 & ? & 1 \\ 0 & ? & 1 & ? \end{pmatrix}, \quad \mathbf{S}_3 = \begin{pmatrix} ? & ? & 1 & 0 \\ 1 & ? & 0 & ? \\ ? & 0 & ? & 1 \\ 0 & 1 & ? & ? \end{pmatrix}$$

$$\mathbf{S}_4 = \begin{pmatrix} 1 & ? & ? & 0 \\ ? & 1 & 0 & ? \\ ? & 0 & ? & 1 \\ 0 & ? & 1 & ? \end{pmatrix}, \quad \mathbf{S}_5 = \begin{pmatrix} ? & ? & 0 & 1 \\ ? & ? & 1 & 0 \\ 1 & 0 & ? & ? \\ 0 & 1 & ? & ? \end{pmatrix} \tag{3}$$

$$\mathbf{S}_6 = \begin{pmatrix} ? & ? & 0 & 1 \\ 1 & ? & ? & 0 \\ ? & 0 & 1 & ? \\ 0 & 1 & ? & ? \end{pmatrix}, \quad \mathbf{S}_7 = \begin{pmatrix} 1 & ? & 0 & ? \\ ? & 1 & ? & 0 \\ ? & 0 & ? & 1 \\ 0 & ? & 1 & ? \end{pmatrix}$$

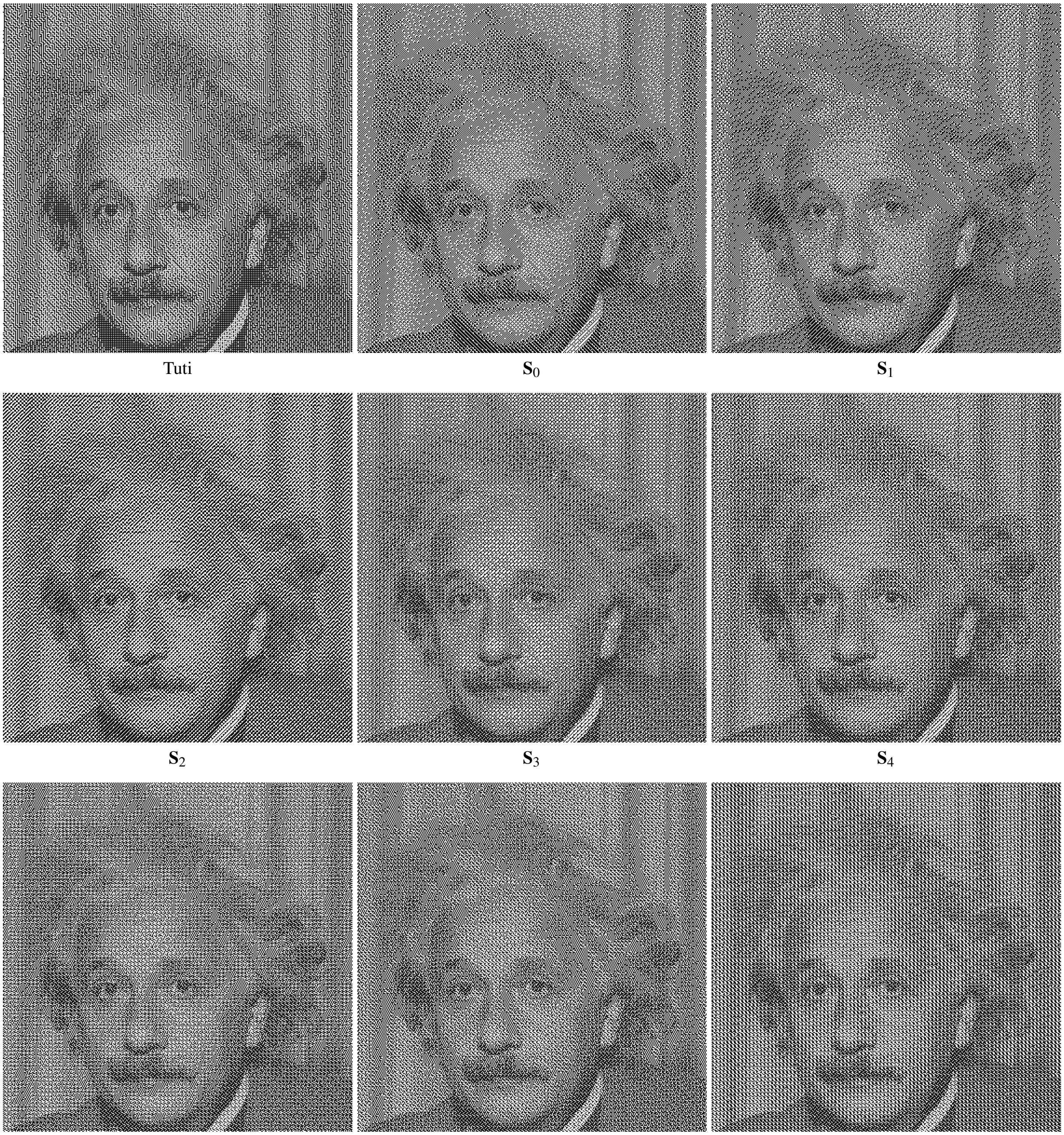
Note that Tuti Interweaving coincides with $\mathbf{S}_2$. For the sake of completeness we also mention a possibility using two-black two-white yarns for the warp and the weft, and the weaving structure coincides with $\mathbf{S}_5$, but this is evidently inferior to Tuti Interweaving, and we will not consider it further.

Feeding the input images for each weaving structure is similar to Tuti Interweaving. We may use a modular approach by listing all the 256 blocks after setting the optional entries, and using them as a 9-grade palette, or we may instead follow a non-modular approach that directly maps to black and white, and gain more granularity in controlling the quantization errors by directly distributing black and white pixels over the 50% set-able entries in the final bitmap. We skip the inferior modular approach in this paper, but the interested reader may refer to [3].

We follow these steps to adapt the Floyd-Steinberg error-diffusion algorithm [9] for direct halftoning: We first adjust the gray levels to account for the 50% loss of contrast. We then diffuse the errors and compute the output pixels the normal way; however, for pixels that are preset according to the chosen weaving structure, we output the preset value, and adjust the errors if the preset output is different from the computed output. In the supplementary material we provide a basic implementation. Unlike Tuti Interweaving, the weaving structure is identical to the visible pattern. The output spatial resolution is the same as the input, but the effective resolution is only 50% of that, since 50% of the pixels are preset, which also implies a 50% global taint of gray. Figure 2 illustrates the output halftone for the eight structures using a given input image.

Disjoint Weaving Structures

In all our previous papers on weaving [1, 2, 3, 4, 5, 8] we were focusing on floats as the only problem with weaving structures. Only recently we noted a serious but subtle problem that may arise with some weaving



Tuti $\mathbf{S}_0$ $\mathbf{S}_1$

$\mathbf{S}_2$ $\mathbf{S}_3$ $\mathbf{S}_4$

$\mathbf{S}_5$ $\mathbf{S}_6$ $\mathbf{S}_7$

Figure 2: *Tuti-like weavable halftonings of an image, showing also the original Tuti Interweaving for comparison.*

structures. Consider, for example, the Tuti Interweaving structure matrix in Eq (2) applied to a black image:

$$\mathbf{S}_{\text{Tuti Interweaving Black}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

Looking only at the matrix, it is not quite easy to spot something wrong with this weaving structure. The problem is seen, however, by understanding how the Tuti Interweaving concept works. To render a black region we send all white yarns behind all black yarns at their intersections, hence we end up with two disjoint fabrics! Weaving practitioners might be familiar with this, but the problem is subtle and not easily discoverable in a mathematical abstraction. While floats are easily discoverable from the weaving matrix (long runs of 0s or 1s), it is not clear if we can have a similar simple rule to discover disjoint structures. It is worth mentioning that these disjoint weaving structures, known as "double weave" or "pocket weave", are sometimes deployed deliberately [7].

Fortunately, we could outline simple steps to trace such a problem in our block-based structures. Suppose that we label the four white warp yarns (left to right) $\{a,b,c,d\}$, and the four black weft yarns (bottom to top) $\{A,B,C,D\}$. Let us use '>' to indicate that a yarn runs above another. Since the template matrix blocks are designed to have each yarn running once above and once beneath, we may trace cyclic chains

$$\begin{aligned}
\mathbf{S}_0 &: \{a > A > b > B\},\{c > C > d > D\} \\
\mathbf{S}_1 &: \{a > A > b > B > c > C > d > D\} \\
\mathbf{S}_2 &: \{a > A > c > C\},\{b > B > d > D\} \\
\mathbf{S}_3 &: \{a > A > b > B > d > D > c > C\} \\
\mathbf{S}_4 &: \{a > A > c > C > b > B > d > D\} \\
\mathbf{S}_5 &: \{a > A > b > B\},\{c > C > d > D\} \\
\mathbf{S}_6 &: \{a > A > b > B > c > D > d > C\} \\
\mathbf{S}_7 &: \{a > A > c > D\},\{b > B > d > C\}
\end{aligned}$$

for the weaving blocks in Eq (3). We see that for $\mathbf{S}_{1,3,4,6}$ the preset weaving structure guarantees that all yarns interweave, regardless of the optional intersections, whereas for the other four blocks there is a chance that we end up with two disjoint fabrics; specifically if the optional entries are such that all yarns in a chain stay above or beneath all yarns from the other chain. It is known when this happens for the original Tuti Interweaving, but it needs further investigation in our extended designs.

Frequency Contents of Weaving Structures

Beyond having different structural properties, Tuti Interweaving, and the eight Tuti-Like variants, generate different results according to how the structure of the fixed weaving pattern interacts with the frequency contents of the rendered image. Figure 2 shows an example using the same image. A typical application for Tuti-like weaving design should offer the user a gallery of the eight designs to choose, since the result may differ from one image to another.

Conclusion

In this paper we presented improvements over the Tuti Interweaving concept we introduced in a previous Bridges paper. It is interesting to see an example of how models could be developed. The original design implied both operational awkwardness and potential technical problems with the generated fabric, but going through that model first is what opened the way for the simplified designs presented here. Now, this paper opens the door for an interesting research topic in computer graphics. Some of the halftones shown in Figure 2 exhibit an evident *aliasing* pattern in some parts of the image, and it would be interesting to study the frequency structure of the eight weaving blocks, and how they interact with the frequency contents of the input images.

It is worth noting that many of our previous papers on monochrome weaving design should be revised for the disjoint weaving structures problem highlighted in this paper.

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