Walkable Curves and Knots

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Abstract

Human-scale walkable structures have been designed, based on fractal curves, spirals, and knots. Traversing such structures in one's imagination is entertaining and can provide a deeper understanding of mathematical objects. If some of these structures were built, for example in a public park, they would help people to better understand and appreciate mathematics. They could also find application in video games. Small models of several of them have been 3D printed.

Introduction

Several of M.C. Escher's most popular prints feature a traversable closed path of some sort, including *Ascending and Descending* (1960), *Waterfall* (1961), and *Reptiles* (1943). Others depict a scene with multiple staircases arranged in unusual fashion, such as *Relativity* (1953), *Convex and Concave* (1953), and *House of Stairs* (1951) [1]. These generally involve optical illusions or unusual perspective systems, and the viewer can't help imagining what it would be like to explore such a world.

This paper describes human-scale walkable structures based on fractal curves, spirals, and knots. These constructs bear some similarity to 3D-printed knight's tours by Robert Bosch [2]. In Bob's words, these sorts of structures can "pull the viewer into the artwork and lead them to make discoveries or to see something in a new light". They speak to our urge to explore new, sometimes enigmatic, and intriguing spaces.

Some real-world architectural structures are resonant of the imagined constructs described below. Stepwells are wells in which the water is reached by descending a series of steps. The finest examples are located in India [3], and some are strongly reminescent of Escher's prints involving multiple flights of stairs. One wonders if Escher was aware of these, as the columns and cupolas seen in buildings like that of Fig. 1a are also similar to features in some of Escher's prints. A contemporary structure that consists almost entirely of stairs and landings is under construction in New York City. "Vessel", with 154 flights of stairs, "is intended to be climbed, explored, and experienced" [4]. There is a rollercoaster-like structure in Germany that can be only be traversed via stairs [5].

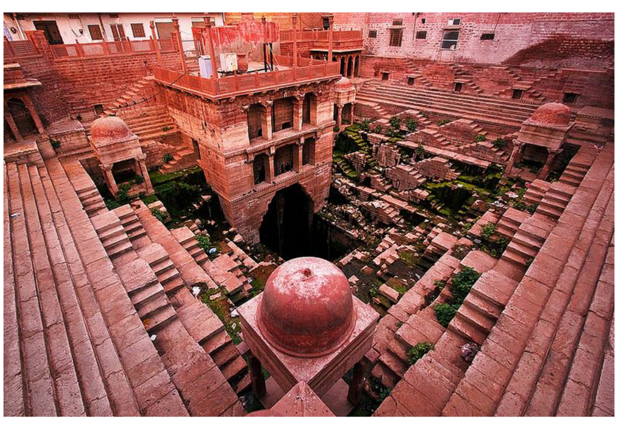


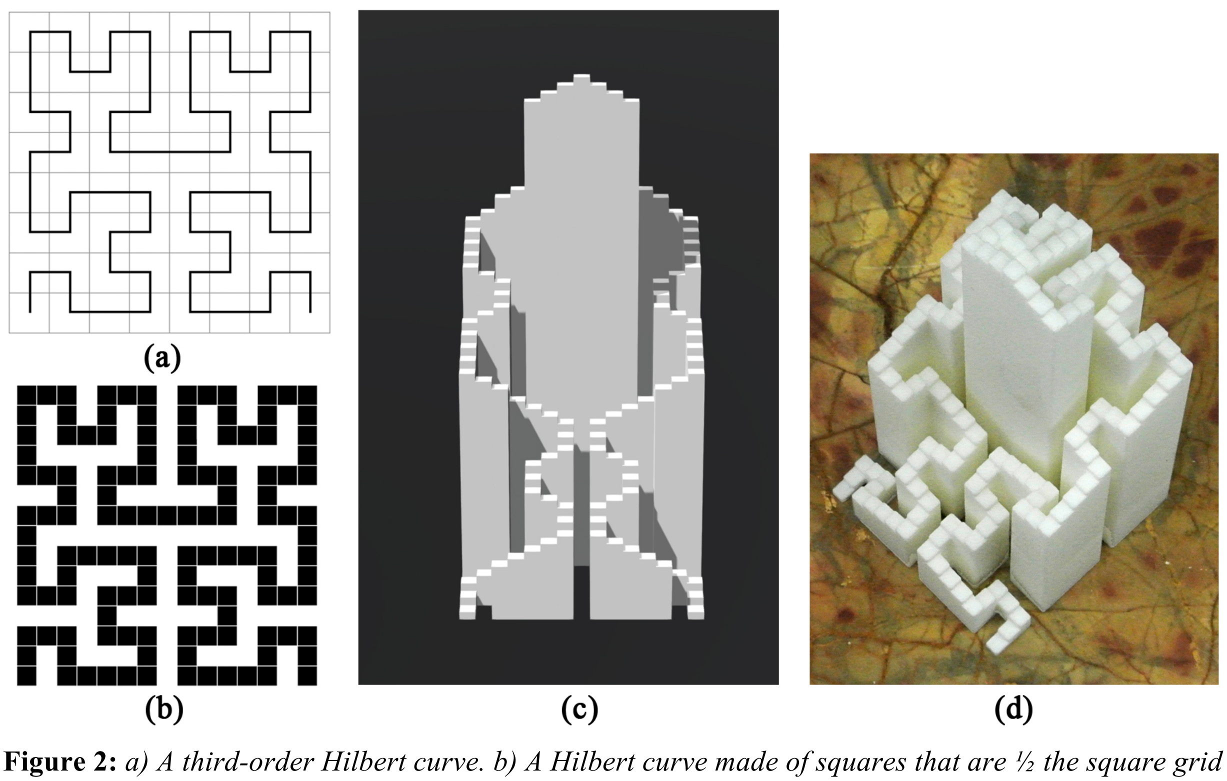

Figure 1: a) A centuries-old stepwell, *Toor ji ka baori*, in Jodhpur, India (photograph by SaraswaT VaruN). b) *Vessel*, a structure consisting of stairs and platforms, under construction in New York (photograph by Mike Peel; www.mikepeel.net).

Fractal Curves

Fractal curves provide convenient templates for walkable mathematical structures. The Hilbert curve is particularly easy to work with due to the face that it's based on a square grid. It's basically an algorithm for connecting the squares in a grid to form a non-intersecting path. This is illustrated in Figure 2a. The curve is covered with smaller squares in Fig. 2b that define a path equal in width to the spacing between the rows of squares. If each of these squares is imagined to be a stepping stone, a walkable "tower" can be constructed (Figs. 2c, 2d). The bilaterally-symmetric stepped form of this structure calls to mind Art Deco skyscrapers like the Empire State Building and Chrysler Building.

The Gosper curve is a plane-filling curve that provides an algorithm for connecting a tiling of hexagons (Fig. 3a). It can be covered with hexagons in a similar manner to that used for the Hilbert curve with squares. This results in a chain of hexagons, a polyhex, with gaps that are long chains of hexagons, as shown in Fig. 3b. If the hexagons are treated as stepping stones, the tower-like structure of Figs. 3c and 3d results. This structure looks much less architectural, due both to the lack of bilateral symmetry and the lack of right angles. If anything, it evokes natural basalt columns such as in the Giant's Causeway.

A third example is based on a path I discovered a few years ago that connects the squares comprising a Sierpinski Carpet [6], shown in Fig. 4a. The large open square in the center of this curve brought stepwells to mind. Making each square in Fig. 3a a square step results in the structure of Fig. 4b. These sorts of broad square steps don't evoke actual stepwells very strongly, so I decided to add small flights of stairs between the large squares, which become landings (Fig. 4c, d).



(a) (b) (c) (d)

Figure 2: *a) A third-order Hilbert curve. b) A Hilbert curve made of squares that are ½ the square grid spacing. c) Rendering of a structure created by stepping up each square to the midpoint of the curve and then back down. d) A 3D print of the structure shown in c.*

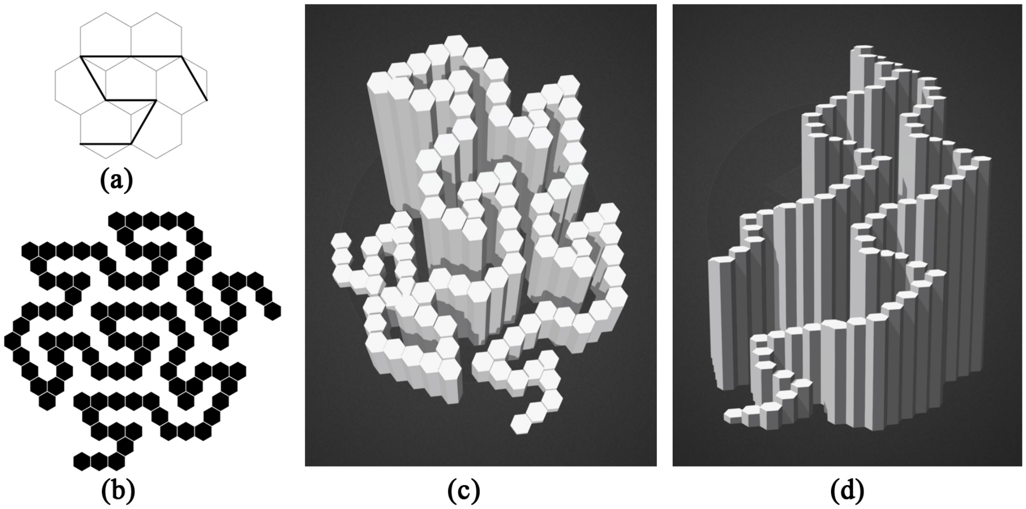


Figure 3: *a) A first-order Gosper curve. b) A second-order Gosper curve made of hexagons. c, d) Renderings of a structure created by stepping up each hexagon to the midpoint of the curve and then back down.*

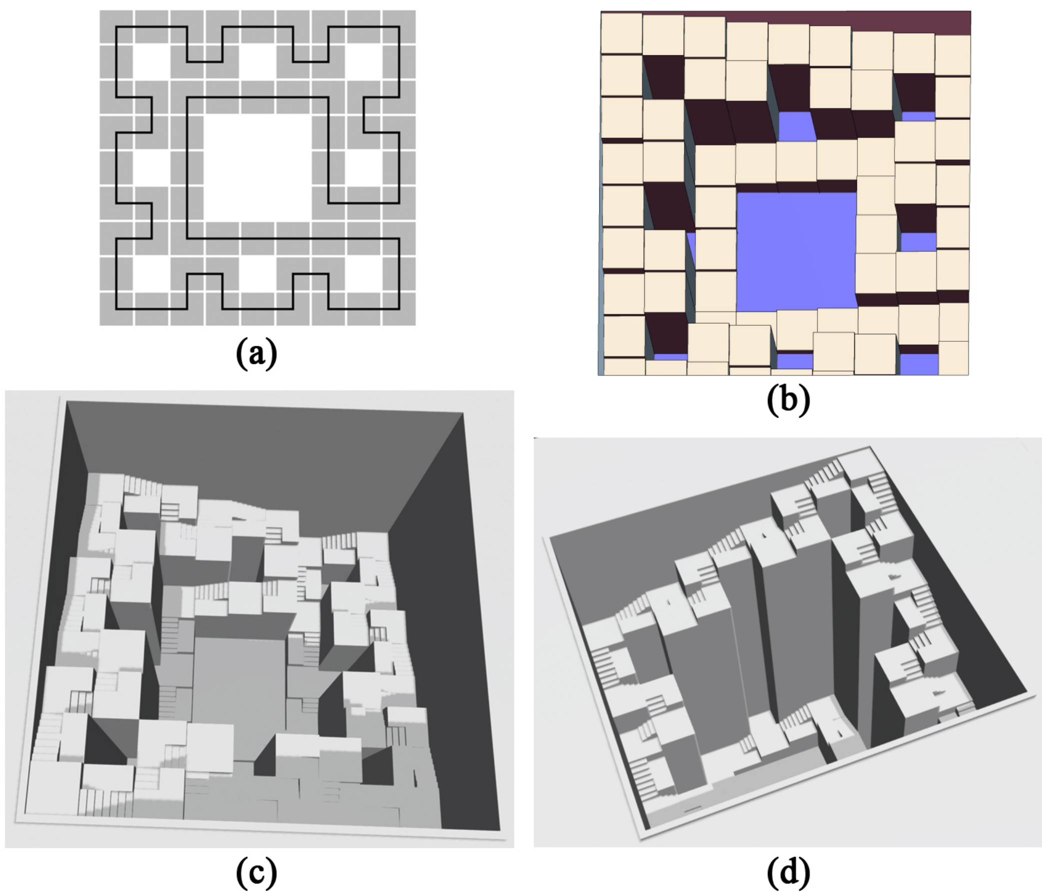
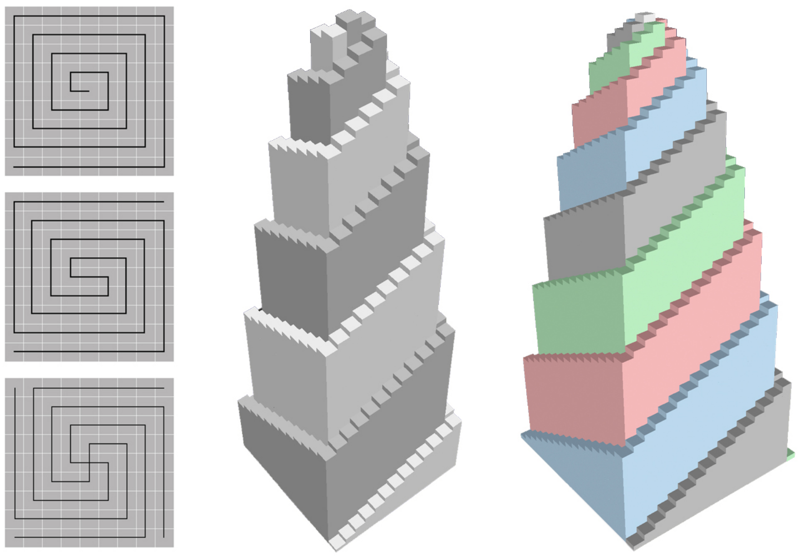


Figure 4: *a) A path that delineates a second-order Sierpinski Carpet. b) A step well created by making the squares of a into square steps. c, d) Renderings of a step well based on b, but with small staircases added between the large square cuboids.*

Spirals

Another way to connect tessellated squares is with a spiral, as shown in Fig. 5a. In addition to single spirals, double and quadruple spirals are possible. Towers created by forming steps from the individual squares are shown in Figs. 5b and 5c. The steps could obviously be replaced by smooth ramps or varied in other manners, as well as the slope of the ramps being varied. It's not hard to imagine a modern skyscraper based on such a form. Similar spirals based on hexagons and triangles are also possible. These can be seen as polygonal analogs of an Archimedean spiral, a circular spiral with even spacing between adjacent turnings.



(a) **(b)** **(c)**

Figure 5: *a) Single, double, and quadruple square spirals. b) A tower created by making the squares of a double spiral into square steps. c) A tower created by making the squares of a quadruple spiral into square steps.*

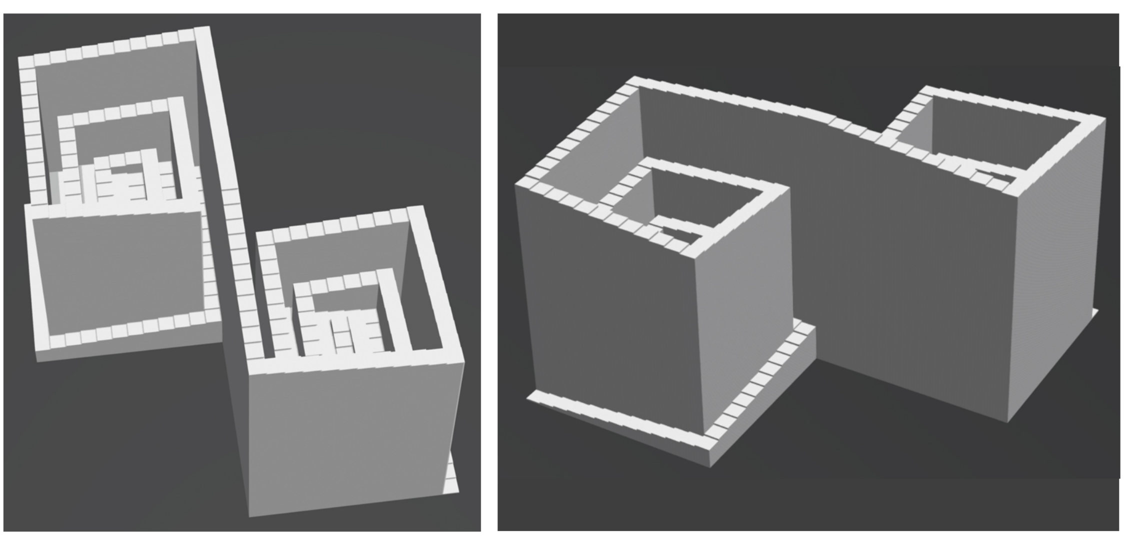


Figure 6: *Two views of two connected square double spirals, forming a single path from one corner to the opposite corner.*

A variation on this sort of form is a double-ended spiral, two finite spirals joined smoothly at the ends. There are also different ways to employ spirals in 3D structures. E.g., a tower made from a double spiral could rise continuously along its path rather than rising to a peak at the center and then returning to ground level. In Fig. 6, this is employed in a double-ended square double spiral. In each half, starting at ground level the ramp rises to a middle height at the center and continues to rise as the spiral progresses back outward. If a person were walking up this ramp, he or she would pass through a narrow deep channel to a relatively level center and then on along a tall narrow ridge.

Spiral helices similar to those in Figure 54 have been used in actual buildings. The Tower of Gor, in modern-day Iran, is based on a square spiral. A 19th-century drawing is shown in Fig. 7a; the building is now a ruin. The minaret of the Great Mosque of Samarra, Fig. 7b, in modern-day Iraq, is a ninth-century structure based on a cicular spiral.

Note that the vertical spacing between successive wrappings of the spiral are relatively uniform in Fig. 7b. As seen in Fig. 5, with uniform ramp/staircase slope the spacing decreases as the peak is approached. This is due to the fact that the amount of material needed to complete a full 360° gets smaller as the peak (center) is approached. Clay offers a convenient medium to create a similar structure to the minaret of Fig. 7b. An elongated right triangle of clay with uniform thickness was rolled to form the spiral helix of Fig. 7c, where the hypotenuse of the triangle provided the constant slope. As in Fig. 5, the spacing of the wrappings is seen to decrease as the peak is approached. In order to achieve more uniform spacing, the slope of the ramp must increase as the peak is approached. This can be observed on close examination of Fig. 7b.

(a) **(b)** **(c)**

Figure 7: *Spiral helical structures. a) The Tower of Gor; b) The minaret of the Great Mosque of Samarra (photograph by Vlastni Photo); c) A clay form with a ramp of constant slope.*

Links and Knots

The crossings in links and knots create challenges and opportunities for more complex and interesting walkable designs. The Borromean rings are commonly shown in two dimensions as three flat interlocking circular rings. It's well known that this structure isn't possible as drawn, with distortion from perfect circles being required in a 3D object. In a walkable structure, bridges allowing one ring to cross over another naturally provide the sort of distortion needed, as shown in Fig. 8. A 3D file that can be manipulated is available, and this design has been 3D printed [7]. The designs in Figures 9 and 10 have also been printed.

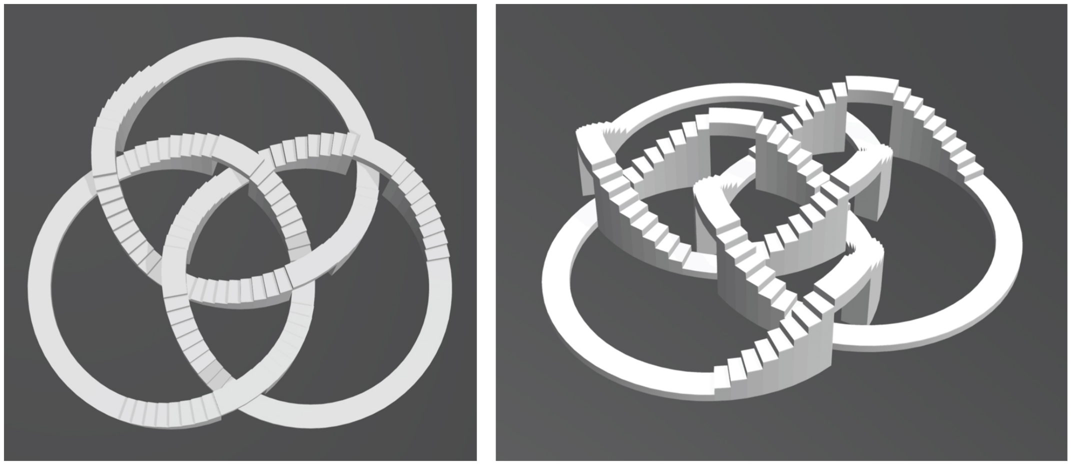


Figure 8: *Three-dimensional model of walkable Borromean rings.*

The five-crossing knot designated 5_1, with five-fold symmetry, is a relatively simple starting point for designing a walkable knot. Since it's an alternating knot, five pairs of up and down flights of stairs suffice to create the five crossings. A pentagram inset in a regular pentagon was employed in the layout of Fig. 9 to emphasize the five-fold nature of the knot. Note this is the torus knot designated T(5,2) [8].

In these designs step dimensions were chosen that are similar to those found in typical relatively narrow staircases; e.g., a step height of 8" (20 cm), depth of 12" (30 cm), and width of 2.5' to 3.0' (0.75 m to 0.9 m). The slope of the flights of stairs is constant, as in real buildings, and the clearance for walking underneath bridges is adequate for a typical person.

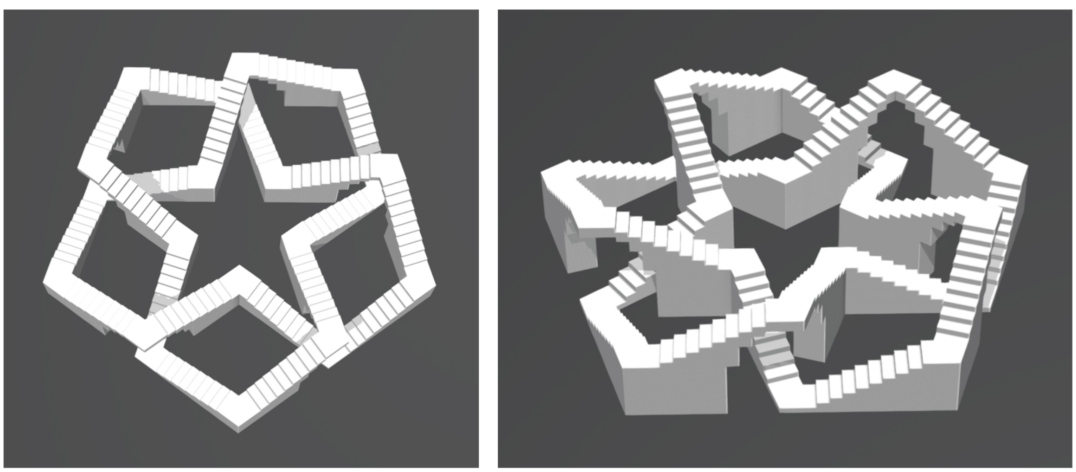


Figure 9: *Three-dimensional model of a walkable five-crossing knot.*

Straightforward walkable designs based on alternating knots will always contain a single bridging level in addition to the ground level. Using torus knots with more crossings, which are not in general alternating, requires more levels. This creates potential for more dramatic designs, such as that shown in Fig. 10. In this case another five-fold knot, the (5,3) torus knot, which has ten crossings, was used. In contrast to Fig. 9, the structure of Fig. 10 has three levels. The design has a distinct Escher flavor to it. Unlike most of Escher's prints, there are no optical illusions or other physical impossibilities here. As a result, such a structure could actually be built in a public space, affording an amusing form of exercise and social interaction. Smoothly-curved arches were added under the bridges in this design, in contrast to the designs of Figures 8 and 9.

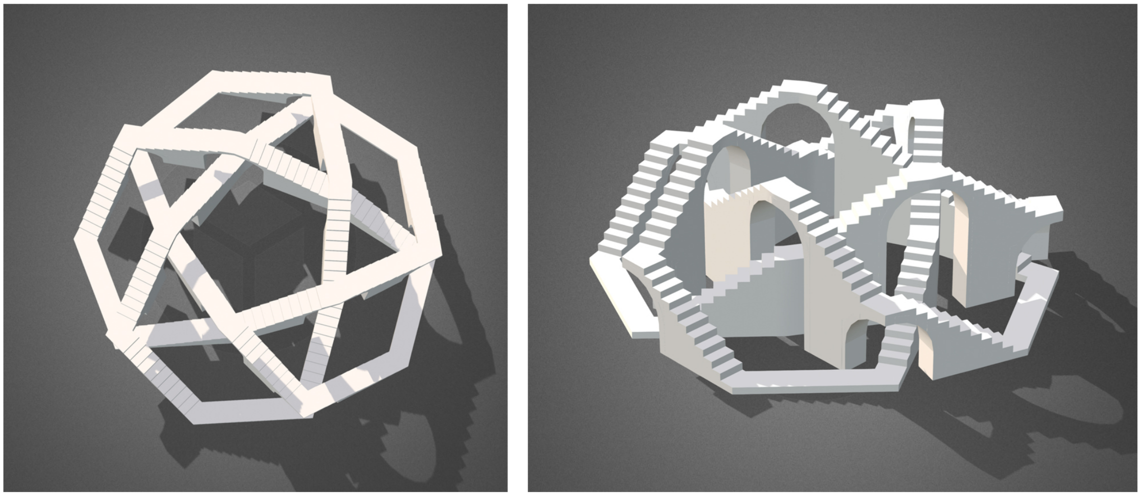


Figure 10: *Three-dimensional model of a walkable (5,3) torus knot.*

With a more complex knot like the (6,5) torus knot, with 18 crossings, it becomes more challenging to fit all the crossings in with bridges that allow adequate clearance. After some initial attempts with stairs and bridges, I chose to use a combination of spiral staircases and slides, as shown in Fig. 11. A spiral staircase allows a large elevation gain in a small footprint. Descending on slides would be more fun than walking stairs, and the slides allow for graceful smooth curves. If it were actually built, this would obviously be a much larger structure than the torus knot of Fig. 10.

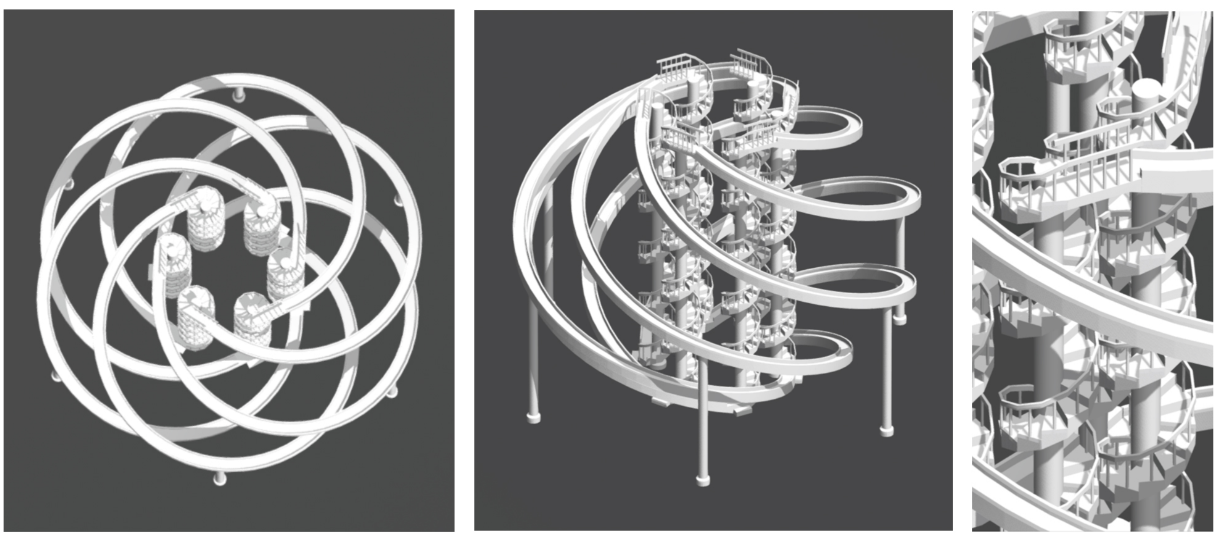


Figure 11: *Three-dimensional model of a walk-up, slide-down (6,5) torus knot.*

In the last example, a nine-crossing knot is used that contains a square grid of alternating weave. This knot layout comes from an iterated knot I designed previously [9]. More architectural detail is added in this example, as seen in Fig. 12. The cupolas supported by pillars are consciously patterned after the sort of architectural details Escher used in prints like Belvedere. Since the crossings are laid out as a square weave, they could obviously be extended to an arbitrarily large structure.

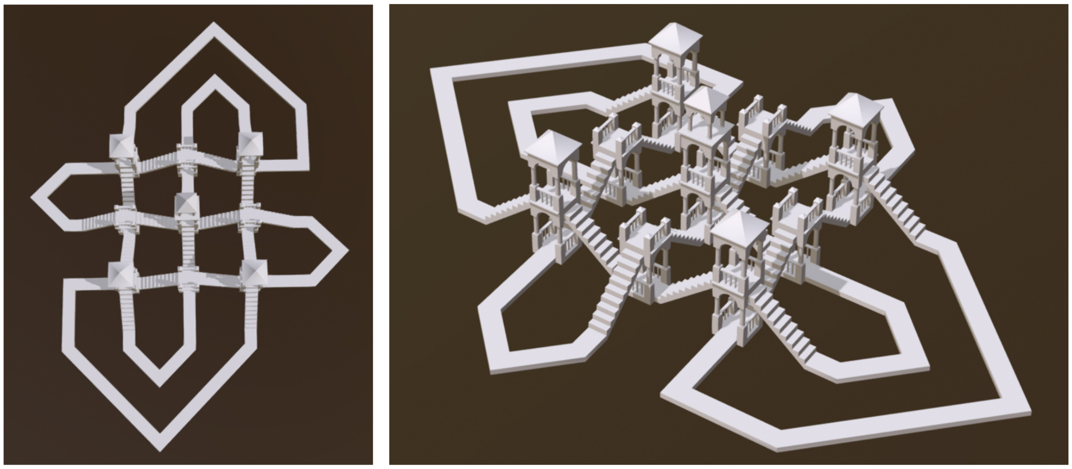


Figure 12: *Three-dimensional model of a walkable nine-crossing knot.*

Summary and Conclusions

Imagined human-scale walkable structures have been designed based on fractal curves, spirals, and knots. The types of structures include towers, stepwells, walking/exercise paths, and slides. It's entertaining to traverse these mentally, and doing so can provide a deeper understanding of mathematical objects. Real-world structures based on these some of these designs could actually be built, and they could also find application in video games or marble runs. One direction for extending this work would be designs with more three-dimensional character, such as a Hilbert curve in three dimensions or polyhedra-based structures. Another would be more topologically-interesting forms such as a Klein bottle. Walkable impossible objects are another possibility. It should also be possible to create a standard set of design rules that would allow automatic generation of a walkable structure from input of any known knot.

References

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