A Personal Approach to the Art of Mathemagic with a Deck of Cards

Colm Mulcahy

Spelman College, Atlanta, Georgia, USA; colm@spelman.edu

Abstract

Mathematics underpins numerous classic amusements with cards, from forcing to prediction effects, and many such tricks have been written about for general audiences by popularizers such as Martin Gardner. The mathematics involved ranges from simple "card counting" (basic arithmetic) to parity principles, to surprising shuffling fundamentals (Gilbreath and Faro) discovered in the second half of the last century. We'll discuss three original and totally different principles discovered since 2000, as well as how to present them as entertainments in ways that leave audiences (even students of mathematics) baffled as to how mathematics could be involved.

Introduction

Magic is one of the performing arts where mathematics can be subtly applied to produce impressive results which conceal the underlying principles. Martin Gardner (1914-2010) was a pioneer in writing about this for the general public, over a period of many decades going back to his classic "Mathematics, Magic and Mystery" book [2] from 1956. Fernando Blasco presented on the topic at Bridges 2011 in Portugal [1].

Over the last ten to fifteen years, the author has been lucky to come up with the three effects explained in detail below. Whether (two and a half of) the three underlying mathematical principles were discovered or created is open to discussion—the remaining half was found in a book from the 1960s. In any case, the effects have proved to be very popular with audiences ranging from 8 to 80+ year olds, when suitably presented.

They appeared in print in the author's book [7]. We start with fresh presentations of them: describing how they appear to audiences, how to perform them, and what mathematics makes them work. We then share new insights on the performance art aspects, all based on years of subsequent experience and development.

Little Fibs

How it Looks: Shuffle a deck of cards a few times, and take off the first five or six cards face down. Mix those thoroughly in your hards, perhaps with your eyes closed for dramatic effect, then have two selections made, one each by two different spectators. They look at and remember their individual cards, share the results with each other, and finally one announces the sum of the chosen card values. Here the usual convention is followed where an Ace is worth 1, a Jack 11, a Queen 12 and a King 13. Pause, and emphasize that the deck was shuffled, you couldn't possibly know what any of the cards were, and so on. Regardless of what total is reported, you promptly announce what each individual card is, value and suit!

What's Really Going On: This sounds too good to be true, e.g., if a total of 14 is reported then it could have been two 7s, or an 2 and a Queen, or one of many more possibilities. Even if you knew which combination it was, the suits bear no relation to the sum of the card values, so how could you know those as well?

That last observation serves as a clue: Totally free choices are indeed offered, but from a carefully controlled small subset of the deck. The possibilities may be narrowed down by having half a dozen key cards at the top of the deck at the start, in any order, and keeping them there throughout some fair-looking shuffles. One way to do that, instance, is with riffle shuffles where you always drop last the original top 8 or more cards, perhaps tilting the top of the deck towards you so the audience's attention is drawn to the bottom cards. It's surprising how easy it it to get away with this. In advance, you need to memorize the suits (as well as the values) of the half dozen key cards which start at the top.

The mathematics which makes the described effect possible is about numbers (card values). Those are the smallest positive Fibonacci numbers: start with 1, 2; add to get the next one. Repeat. Hence we get, $1 + 2 = 3, 2 + 3 = 5, 3 + 5 = 8, 5 + 8 = 13$, and so on. The list of Fibonacci numbers continues forever: $1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, \ldots$. However, for magic with playing cards, we focus on the first six of these, i.e., 1, 2, 3, 5, 8, and 13, agreeing that 1 = Ace and 13 = King. Now, consider some particular cards with those values, for instance, A♣, 2♡, 3♠, 5♢, 8♣, K♡ (CHaSeD order), in other words:

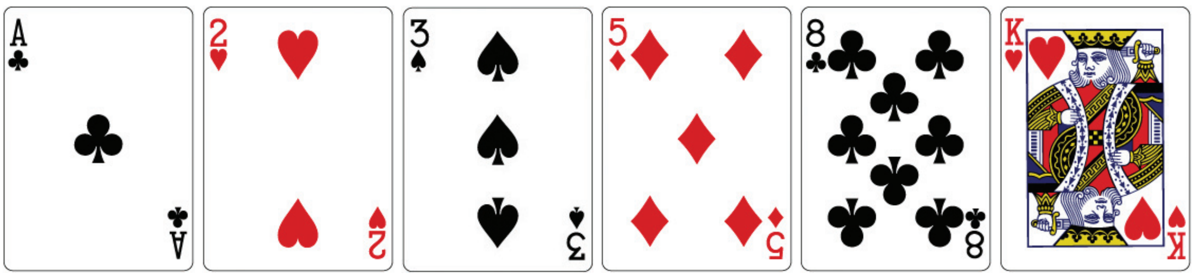

Figure 1: *Little Fibs.*

These are the **Little Fibs**—something you can draw attention to later if explaining the effect to your audience, remarking that you did tell a few little fibs at the outset ("the cards have been totally shuffled," etc).

Why it Works: If any two cards are selected from these special ones, then they can be determined from their value sum, because of the *unaddition property*: **Any possible total can only arise in one way.**

That tells you the values of the chosen cards; the suits you have memorized. Of course you don't know which card is which, but don't draw attention to that. Sometimes people will mistakenly remember later that you knew which card belonged to which person!

It's easy to break up the sum into the two Fibs it's made up from: just peel off the largest possible Fib, what's left is the other one. (This is a kind of greedy algorithm.) For instance,

$$8 = 5 + 3,$$
$$10 = 8 + 2,$$
$$14 = 13 + 1,$$
$$18 = 13 + 5,$$
$$\ldots$$

In practice, we suggest ditching the Ace and only allowing free choice from the top five cards. Thus a total of 3 can't arise; it's no surprise that you know the original cards were an Ace and a 2 in that case.

This is not an effect that should be done twice for the same audience, because of the risk of one or two of the same cards showing up in a second performance. However other sequences of five numbers can be used, and if you are feeling brave by starting with ten special cards at the top and keeping them there throughout some false shuffles, you can repeat the effect once and baffle your audience even more.

What other lists of numbers work? The Lucas sequence for one, that starts 2, 1, 3, 4, 7, 11, 18, ..., where each term is once more obtained by summing the previous two. It's a kind of generalized Fibonacci sequence. If we omit the 2, it too has the desired "unaddition" property. (You can never use a Ace, 2, 3, and 4 at the same time, because $1 + 4 = 2 + 3$.) We don't even need generalized Fibonacci sequences. The numbers 1, 2, 4, 6, 10 work. So do 1, 2, 5, 7, 13.

Source: This was conjured up in January 2008—in the middle of teaching a class about the Zeckendorf representation—and was first published as "Additional Certainties" the following month in the bi-monthly **Card Colm** column or blog at MAA.org [6]. It can be found in Chapter 4 of the author's 2013 book [7]. It's also featured in a 2013 *Spelman College* video [9] and a 2016 *Numberphile* video [10].

The Three Scoop Miracle

How it Looks: Hand out the deck for shuffling. A spectator is asked to call out her favorite ice-cream flavor; let's suppose she says, "Chocolate." Take the cards back, and take off about a quarter of the deck. Mix them further until told when to stop. Deal cards, one for each letter of "chocolate," before dropping the rest on top as a topping.

This spelling/topping routine is repeated twice more—so three times total. Emphasize how random the dealing was, since the cards were shuffled and you had no control over the named ice-cream flavor. Have the spectator press down hard on the final top card, asking her to magically turn it into a specific card, say the 4♠. When that card is turned over it is found to be the desired card.

What's Really Going On: This time, unlike in the first effect above, the deck really is shuffled at the outset. You don't need to know the top five or six cards.

The card effect as described works because, despite the genuine shuffling done at the start, you get a peek at the bottom card before the spelling and dropping begins. The mathematics takes care of everything else: at the end you casually ask the audience member to turn the top card into the one you saw earlier (and remembered!).

The key move here is a *reversed transfer* of a fixed number of cards in a packet—at least half—from top to bottom, done three times total. "Chocolate" (9 letters) requires a packet of between 9 and 18 cards, ideally 11 to 16, hence the "about a quarter of the deck" suggestion. "Vanilla" is only 7 letters, so 7 to 14 cards are required, ideally 9 to 11. A flavor such as "mint chocolate chip" (17 letters) requires a lot more cards and a lot more spelling, the latter making it a little tedious given that it has to happen three times over. Of course, "mint" or "mango" are so short that the packets feels too small. Since most people say vanilla, chocolate or strawberry, in our experience, it's not a bad idea to start with 12 cards in your hand and adjust only if necessary.

Let's try to get organised. The dealing out of k cards from a packet that runs $\{1, 2, \ldots, k-1, k, k+1, k+2, \ldots, n-1, n\}$ from the top down, and then dropping the rest on top as a unit, yields the rearranged packet

$$\{k+1, k+2, \ldots, n-1, n, k, k-1, \ldots, 2, 1\}.$$

This is hardly inspiring, or revealing! However, it turns out that when $k \geq \frac{n}{2}$, then *doing this three times in total brings the original bottom card(s) to the top!* This we call **The Bottom to Top Property**.

It seems like magic, and people of all ages almost invariably love it. Don't be surprised if people gasp and ask, "How did you do that?" The answer to which, of course, is a gracious, "Very well, thank you."

Why it Works: There are several ways to analyze the mathematics of this dealing and dropping, as well as what happens if it's done one more time. We offer two explanations here.

Given a packet of n cards and $k \geq \frac{n}{2}$ (arising from a flavour of length k and a number $n \leq 2k$) the packet breaks symmetrically into three pieces (or subpackets) T, M, B (top, middle and bottom) of sizes $n-k$, $2k-n$, $n-k$, such that the count out (of k cards each time) and drop (the rest) move is effectively this operation:

$$T, M, B \to B, \overline{M}, \overline{T},$$

where the bar indicates complete reversal of a subpacket. Using this approach, the Bottom to Top (with three moves) property can now be confirmed easily. Actually. . . **the Bottom to Top Property** is only 75% of the story, as one more count-out-and-drop confirms. Here's the real scoop:

The Period 4 Principle: If four count outs of k cards (dropping the rest on top) are done to a packet of size n, where $k \geq \frac{n}{2}$, then every card in the packet is returned to its original position.

This too can be verified easily enough, either with actual cards, or symbolically, using the $T, M, B \to B, \overline{M}, \overline{T}$ formulation four times over.

Here's a second, more visual, argument: a veritable "proof without words." One can follow along using a packet consisting of all the cards from one suit, for instance, first arranged in order from top to bottom.

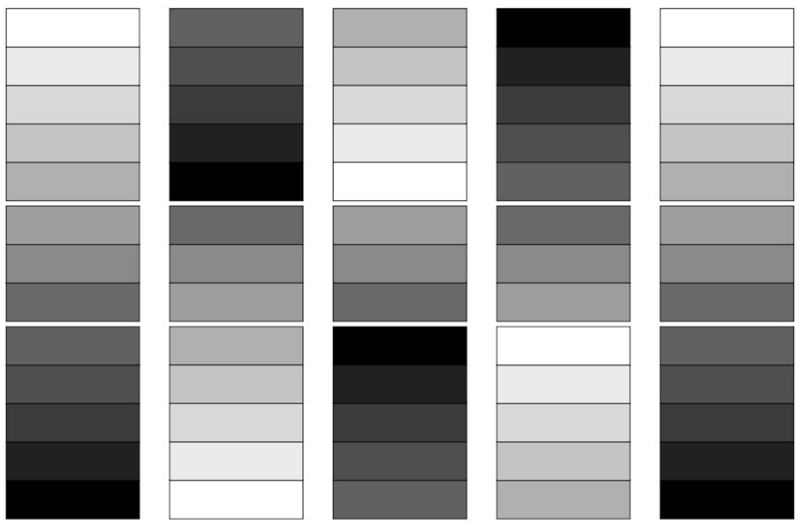

Figure 2: *Proof without words.*

The vertical strip on the left represents 13 cards (from Ace to King) whose numerical values are indicated by gradually darkening shades of gray: white for the Ace on the top down to jet black at the bottom for the King. The second vertical strip represents the result of counting out $k = 8$ of those $n = 13$ cards from the top, hence reversing their order, and dropping the remaining 5 on top. This is what would happen, for instance, working with 13 cards and the flavor "mint chip" (which we find is often mentioned by Californian volunteers). The original bottom card—the King—is easy to track as it is the darkest image. The middle vertical strip shows the situation after two stages of the effect. The fourth vertical strips confirms that the original bottom card is on the top at this stage, which is why the ice cream effect works. The final vertical strip confirms that after four count-out-and-drops, we are indeed back to where we started. (Note that the T, M, B parts are separated in the images by dotted horizontal lines, and have sizes $5, 3, 5$, respectively.)

This period 4 property, incidentally, suggests an in-hand false shuffle: hold a packet of about a dozen cards in one hand, face down, and casually shuffle off exactly 7 of them, one by one, into the other hand, thus reversing their order. Then drop the rest on top. Remark that you need to mix up the cards, while blatantly doing all of that three more times. The cards are now back in the order they started in, which could be very useful if you need them in a particular order for some effect!

Source: This piece of fun was conjured up in a Madrid suburb in early 2003, and was first published on 21 October 2004 at MAA.org as "Low Down Triple Dealing" and dedicated to Martin Gardner on the occasion of his 90th birthday [3]. (That was the first *C*ard Colm, a bi-monthly column or blog which ran for 10 years.) It can be found in Chapter 1 of the author's 2013 book [7]. It's also featured in a 2013 *Spelman College* video [8] and a 2016 *StandUpMaths* "Quick Mathematical Card Trick" video [11].

Poker Hand Control

How it Looks: Hand out the deck to a spectator for shuffling, the more jumbled it is the better. Take it back, and deal out two piles of five cards in an alternating way, remarking, "Let's deal out two poker hands." Then announce, "In a moment I'm going to let you decide who gets which cards. I just want to take a peek at what we have here."

Pick up one hand of cards, glance at the faces, then pick up the other hand, and look at those faces, tucking one hand of cards behind the other. Mutter vague (true!) things about what you see, such as, "Interesting, a pair of 4s and a Jack, Queen and King." Then turn the ten card packet face down and declare, "As I said, *you* get to decide who gets which cards."

Holding the cards face down in one hand, take the top two off and say, "One of these is your hole card, the other is mine. Which do you want, top or bottom?" Whichever card is claimed, place it face down in front of the spectator, and place the other face down in front of yourself. This is a good time to interject, something like "Did I mention you'd get the pair of 4s? Sorry, maybe I shouldn't have said that, I wouldn't want to influence your decisions in any way!"

Hold up the next two cards from the packet, and ask, "Which do you want, top or bottom? The other goes underneath." Whichever card is claimed, add it to the spectator's face-down pile *and tuck the other one underneath the packet in your hand!* There has been a subtle change in procedure here, and it is repeated five more times, by which time the spectator may have forgotten the different nature of the first choice offered. Once more, hold up the next two cards from the packet, and ask, "Which do you want, top or bottom? The other goes underneath." Act accordingly. The spectator now has three cards in her pile, as do you in yours.

Then say, "You also get to pick which cards I get; it couldn't be more fair." Hold up the next two cards from the packet, and ask, "*Which one do I get*, top or bottom? The other goes underneath." Whichever card is indicated, add it to your own pile and tuck the other one underneath the ever-shrinking packet in your hand. Do the same thing one more time. At this stage, you have four cards in your hand and each pile on the table has three cards in it. Say, "Back to you," as you once more offer the spectator one of the top two cards, placing the other underneath. Then say, "One more for me," as the spectator selects a fourth card for your pile from the top two of the three left in the packet in your hand.

Tuck the other underneath the sole remaining card and casually say, "One final one for each of us," *putting the top one on the spectator's pile and the other one on yours.* (On rare occasions, the spectator will protest, "How come I didn't get to chose there?" Simply respond jovially, "You had lots of free choices, and I have no idea what those cards are anyway. We always do the last two cards that way."

Now have the spectator pick up her five cards and look at their faces, as you do likewise with yours. Ask, "Did you get those two 4s like I predicted?" Have her display her hand face up, she will indeed have the cards mentioned. Congratulate her, and add, "Unfortunately, you gave me a pair of 9s, so I guess I win this time!" Throw your cards down for all to see.

Similar results can be obtained with the next ten cards from the deck, and in our experience the spectator is usually happy to give it another try. If you wish to be nice, you can let the spectator win this time, and even predict that outcome. It's quite baffling, you seemingly controlling two poker hands with ten random cards, while the spectator makes almost all of the decisions.

What's Really Going On: Two unrelated concepts make this effect possible. First, **The Birthday Card Match Principle,**

Given ten random cards from a regular 52-card deck, then it's very likely (98%) that there are at least two cards of the same value among them.

This says that most of the time, there will be at least a pair (in the poker sense) somewhere in the two hands first dealt out. If this isn't the case, simply show everyone the faces of all ten cards, laughing, "These won't be much fun for playing poker with, please shuffle again." The chances of two such failures in a row is tiny (2% of 2%).

The Birthday Card Match Principle holds for the same kind of reason that given 50 randomly selected people, the chances of at least two of them sharing a birthday (day and month, not year) is 97% (this is the famous Birthday Match Paradox). Of course they *could* be all different, just as your lottery ticket *might* be the winning one, but the chances are heavily against that. For $k \leq 13$ randomly selected cards from a standard deck, the chances that they are all different in value is:

$$\frac{52}{52} \times \frac{48}{51} \times \frac{44}{50} \times \ldots \times \frac{52-4k+4}{52-k+1}.$$

When $k = 10$ this is about 0.02, or 2%, so that chances that they are *not* all of different value is 0.98 or 98%. The upshot is that most of the time, with ten cards, there is at least one pair matching in value. Try it with ten random cards: surprisingly, you'll often get two or three matching pairs, or three of a kind, or better.

The second thing that makes this effect possible is the seemingly fair "spectator's decisions" which are actually predetermined from the start. This allows you to take advantage of the good things which we just saw are very likely to happen. To put it bluntly, if you subtly rearrange the ten cards in your hand when you look at the faces of the two poker hands—yes, right in front of everyone watching—you can guarantee who will win, and even specify in advance (if you wish, it's optional) how that victory will be achieved.

Let's take a closer look at that "fair method" of dividing up the ten cards. Recall that the first choice is a genuinely free one, but it turns out that the next six choices are rigged. Forget the top two cards, and focus on the other eight beneath them. What this effect uses here is **Bill Simon's Sixty-Four Principle** (borrowed from Bill Simon's 1964 book *Mathematical Magic* [12], specifically the effect there called "The Four Queens"):

It is possible to give the illusion of multiple free choices to a spectator, while controlling how to split up a packet of eight cards into two piles of four.

Control of the split happens this way: the bottom four cards always end up in your pile, the spectator getting the top four! When you glance at the faces of the ten cards dealt, assuming that there is at least one matching pair, ensure that the pair of highest value (or the winning cards in general) get moved to be among the bottom four. With a little practice this piece of blatant skulduggery can be mastered soon enough. Sometimes splitting the packet of ten into two parts (not necessarily of five cards each), one part in each hand as you pull them apart while saying something distracting and simply switching the order of the parts upon reassembly, does the trick. Recall the top two cards are out of your control, so make sure neither is key to determining who wins: you only control four of the spectator's cards (likewise your own), not all five.

Why it Works: To understand how the Bill Simon principle works, forget about poker and imagine instead a packet of eight cards consisting of four red cards on top of four black ones. This replaces the "four cards the spectator gets" on top of "four cards you get" scenario. A perusal of the table below reveals the state at various stages of three things: the packet in your hand, the first pile (spectator's) and the second pile (yours).

The "Start" here ignores the first two cards from the original packet of ten. The rows after that correspond to the situation after 1, 2, . . . , 6 "free choices" have been made by the spectator. The illusion of those six free choices—and the invariable fair-seeming end that you casually force without comment—leads to the same outcome every time: the spectator gets the four red cards (in some order, it doesn't matter which) and you get the four black ones (again, in some order).

Stages \ Cards	Packet	Pile 1	Pile 2
Start	R R R R B B B B		
After 1	R R B B B B R	R	
After 2	B B B B R R	R R	
After 3	B B R R B	R R	B
After 4	R R B B	R R	B B
After 5	B B R	R R R	B B
After 6	R B	R R R	B B B
End		R R R R	B B B B

To recap: in practice, given ten random cards, you seek winning ones and ensure they are in the bottom four, knowing that the spectator will get the four above those. Who gets which of the top two cards is outside of your control, so make sure they don't change the balance of power. This is truly a case of "If you play your cards right, you can ensure that later on, you win" and it all happens right under the spectator's nose.

Source: This entertainment was conjured up in 2005 or 2006, and was first published in June 2006 as part of "Better Poker Hands Guaranteed" in the bi-monthly **Card Colm** column or blog at MAA.org [5]. That in turn depended on "Bill Simon's Sixty-Four Principle" two months earlier at the same site [4]. It appeared in print in Chapter 3 of the author's 2013 book [7]. We were certainly inspired by Dave Solomon & John Bannon's stunning effect "The Power of Poker" (which uses a fixed packet of ten cards, not a random packet).

The Art of the Presentation

Performers will be most effective with their own individual approaches to the art of presentation. Here we discuss patters we have found effective over the years. Note that only the first effect requires any kind of set-up, so always do that one first, then the pressure is off and you can relax for any follow-up effect.

Little Fibs: Some kind of false shuffle has to be mastered as you must retain the top stock of five or six cards in place while appearing to jumble the cards. Next, before proceeding, it's a good idea to flip the deck over and show several pairs of card faces from the middle, so audience members know how to add their values, making sure you include examples with Aces (value of 1) and face cards (values 11, 12 and 13).

After the two selected cards have been inspected and memorized, we sometimes have them plunged face down in two random locations in the rest of the deck, have the deck shuffled more. Taking it back, we truthfully comment, "The cards are now totally mixed up. But neither of you is wearing gloves." This often results in the two inspecting their hands sheepishly while the audience laughs. "So your fingerprints, not to mention your DNA, are over the cards you picked. That makes them much easier to find." You then go through the cards face up, squinting dramatically, and pick out their two selections without hesitation.

Furthermore, when the card value total is reported, you can have a little fun, musing, "The values add up to eleven you say? Ok, so it might be a ten and an Ace, or two five and a halfs, or a King and a minus two. There are so many mathematical possibilities." Then proceed, as above.

Three Scoop Miracle: The first key point is to get a peek at the bottom card of the packet which is dealt and dropped over and over. Once the bottom card is known, it's even possible to appear to jumble the packet repeatedly, in hand, while keeping that key card in place, saying, "Tell me when to stop mixing."

Three deal-and- drop stages can be tedious for viewers, so to break it up we often say at the conclusion of the first round, "Don't worry, it's low calorie." Many will chuckle as you do the second round, at the end of which you can say to the ice cream lover, "Have you done magic before?" Most will say, "No," to which you can say to the crowd, "She has never done magic before! But I have a feeling she is about to do so. If she gets this right, please give her a big round of applause at the end."

After the final round, and your invitation to turn the top card into the one you know it has to be, step aside. Point at the top card, say, "Press down hard on it. Concentrate. Turn it into the . . . " Then have it turned over and shown to everybody. The spectator will be pleased to get an enthusiastic round of applause.

Poker Hand Control: You need to be able to do some brazen and level-headed in-hand card rearrangement while chatting about what kind of poker hands might later arise. Your focus should be on figuring out which two or three cards need to be positioned within the bottom four of the ten, without moving anything yet, while your patter gets the audience's attention. The actual move(s), if required, can be done at the end of your blather, usually by separating the cards into two smaller packets and rejoining them the other way around.

This is an effect which can be done over and over with the same person or group of people. Sometimes you win, sometimes you let the spectator win. The key is to predict something about the outcome, verbally or even in writing. Earlier we suggested the gag of correctly predicting the spectator's hand while not mentioning that yours would be better. Vary the approach and patter here to suit the people you are working with. Let them win every time if you like, by ensuring that the winning cards are in positions three to six from the top of the ten-card packet, repeatedly congratulating them on their amazing poker skills.

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