Kaleidoscopes for Non-Euclidean Space

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Abstract

Kaleidoscopes repeat a small part of an input image to make periodic images. In a flat Euclidean plane and using mirror symmetry at straight lines we can only have two-, three-, four- and sixfold rotational symmetries. With circle arcs instead of straight lines and inversion in circles instead of mirror symmetry we can freely choose rotational symmetries. This gives us Poincaré disc representations of periodic images in hyperbolic space or stereographic projections of repeated patterns on spheres. I present an efficient iterative computational method for creating such images.

Introduction

Kaleidoscopes are made of three plane mirrors that form a triangular prism. Looking through one end you see that the triangular image at the other end is reflected and multiplied. This creates a periodic image covering the entire plane. Only a few different kaleidoscopes exist. It is said that you cannot have kaleidoscopes with periodic 5 or 7-fold rotational symmetry. But is that really true?

Rosettes

We first look at rosettes because kaleidoscopic images are combinations of rosettes. To create a rosette we put two mirrors together at an angle. Their reflections make rosettes such as the one shown in Figure 1. It is made of a piece of an input image in the sector between the mirror lines together with four rotated copies and five rotated mirror images. These ten pieces cover a full circle of 360° because the mirrors meet at an angle of 36°. In general, we want a rosette with a k-fold rotational symmetry. This requires $2k$ image pieces and thus the angle between the mirrors has to be $360°/(2k) = 180°/k$.

How do we program a computer to create such images? Obviously, the computer cannot cut out image pieces and glue them together. Instead, we have to find a colour value for each pixel of an output image. To do this we map the position of an output pixel into the sector between the mirror lines using the two mirror symmetries. Then we read the colour of the input image at the mapped position. This determines the colour of the output pixel. It would be unnecessarily complicated to get the mapping directly from the two mirror symmetries. Instead we better use the fact that the two mirror symmetries generate a dihedral group which combines a k-fold rotational symmetry and k mirror axis. You can discover the elements of this group in Figure 1. This group describes all possible mappings due to the two mirror symmetries.

Let us choose a coordinate system with its origin at the intersection of the two lines and an x-axis that coincides with the lower mirror line. A point with polar angle ϕ lies in the sector between the mirror lines if $0 < \phi < 180°/k$. For points lying outside the sector we use in a first step the k-fold rotational symmetry of the dihedral group. It makes the rosette image be a periodic function of the polar angle ϕ with a period length of $360°/k$. Thus we can replace the ϕ by the rest of its division by $360°/k$ setting $\phi = \phi - (360°/k)\lfloor\phi/(360°/k)\rfloor$. Then $0 < \phi < 360°/k$. To program this calculation you should use the universal floor function and not the modulo function, which might give strange results for negative ϕ. A second step is only needed if $\phi > 180°/k$. Then we use that the dihedral group has a mirror symmetry at a

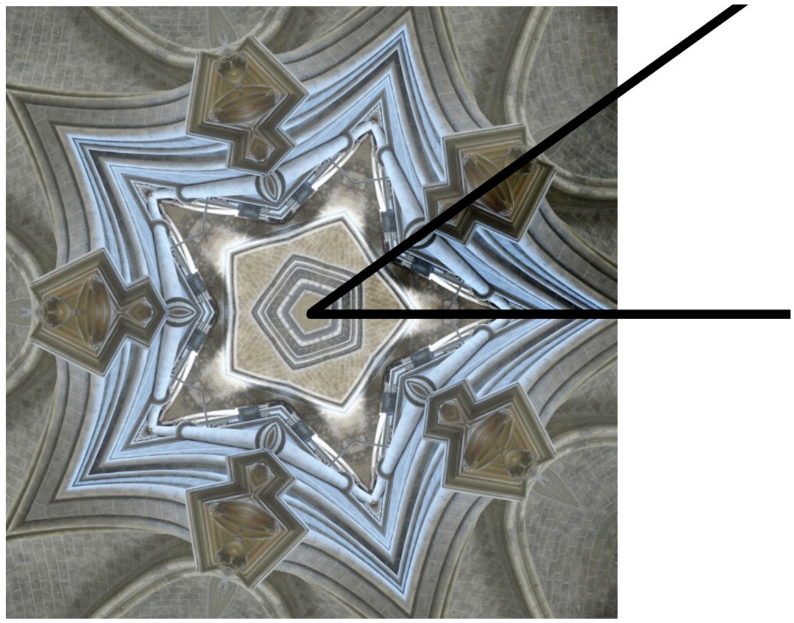

Figure 1: *A mirror symmetric rosette of five-fold rotational symmetry. The black lines show the position of the mirrors.*

polar angle of $180°/k$ and put $\phi = (360°/k) - \phi$. These two steps map any point into the small sector and it becomes easy to create these rosettes. But in practice we have to evaluate trigonometric functions, which can take a lot of computer time. You should speed them up using interpolated function tables [14].

Kaleidoscopes Using Triangles

A triangle of three mirrors makes a kaleidoscope. At each corner arises a rotational symmetry from multiple reflections at two mirrors. Thus every corner is the center of a rosette and the corner angles have to be unit fractions of 180°. We can characterize a triangle by three integers $(k\ m\ n)$. The angles at the corners are $180°/k$, $180°/m$ and $180°/n$. They are creating k, m and n-fold rotational symmetries. Such triangles are known as Möbius triangles, which are special cases of Schwarz triangles. The mirror symmetries at the sides of the triangle generate a triangle group [8]. Applying its elements to the triangle we get a tiling of the plane [7]. Thus we have a unique mapping of output image pixels into the triangle and to pixels of an input image. Yet we need an efficient method for doing it.

The resulting images depend on whether the sum of the angles is smaller, equal or larger than 180°. If we use straight lines for the triangle in the Euclidean plane of the output image, then this sum has to be equal to 180° and thus $(1/k + 1/m + 1/n) = 1$. This equation has only three essential solutions: (4 2 4), (3 3 3) and (6 2 3). As kaleidoscopes they make well-known periodic images with 2, 3, 4 and 6-fold rotational symmetries. You cannot have other rotational symmetries using three plane mirrors put together as a prism.

Kaleidoscopes with a Small Sum of Angles

What happens if we increase the numbers k, m or n? The sum of the angles then becomes smaller than 180°. As an example, (5 2 4) give us angles of 36°, 90° and 45° with a sum of 171°. Obviously, the triangle can still have two straight lines meeting at an angle of 36°. This in itself already makes the resulting image a rosette of five-fold rotational symmetry. The third line cannot be a straight line. Using instead a circle arc we can get corner angles of 90° and 45° generating 2-fold and 4-fold rotational symmetries. Figure 2 shows the triangle and a typical result. For the reflection in the arc we use an inversion in its circle instead of a mirror symmetry. Note that this inversion is a purely mathematical imaging and not the same as an optical reflection in a convex mirror. Yet, there are similarities. Very close to a convex mirror we get the same image as for a plane mirror. The inversion of a point close to the inverting circle gives the same result as a mirror image at a tangent to the circle. Far away the images are reduced in size for both the inversion circle and the convex mirror. Going towards the border of the image we get a decreasing geometric series for the size of the copies of the central motive.

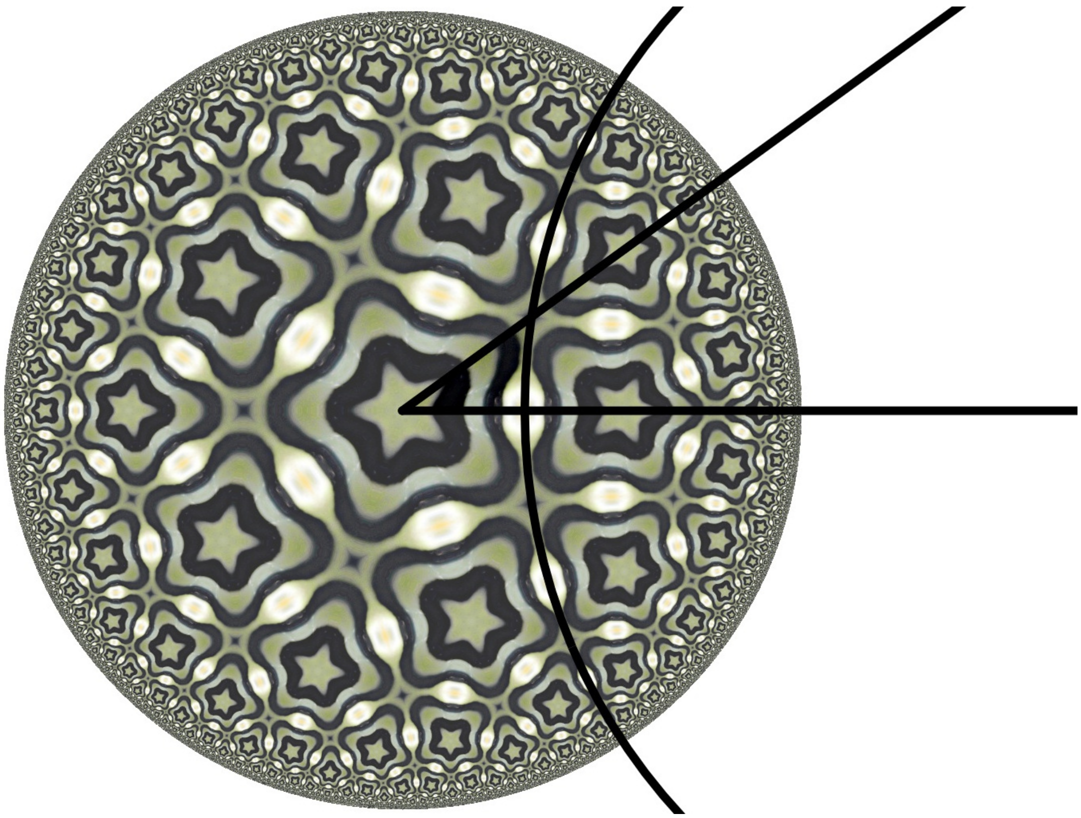

Figure 2: *Kaleidoscopic image with 5-fold, 4-fold and 2-fold rotational symmetries. The black lines show the reflecting sides of the basic triangle.*

This image is actually a Poincaré disc model of hyperbolic space. The arc as the third side of the triangle is a straight line in hyperbolic space as every other circle that intersects the border of the image at a right angle. An inversion in these circles is equivalent to a mirror symmetry at straight lines in hyperbolic space. There it maps straight lines onto straight lines. In hyperbolic space we thus have a kaleidoscope made of a triangle with straight edges that generates a periodic image. The disc model projects this periodic image with a strong compression towards the border of the disc. With an increasing magnification of a very small region at the border we obtain a transition to the Poincaré plane. For more details see Christensen [1, 3], Magnus [8] and Conway [4]. I have published other images of high resolution my album "kaleidoscopic" [11]. In "The Symmetry of Things"[4] you find a detailed discussion of periodic images in hyperbolic space, including a discussion of M. C. Escher's famous "Circle Limit IV".

Inversion in a Circle

A reflection in a straight line is simply a mirror image. As straight lines are circles of infinite radius there should be something like a reflection in a circle that becomes the usual mirror image for very large circles. This actually is the inversion in a circle.

The inversion in a circle of radius R with its center at position $\vec{c}$ transforms a point at $\vec{p}$ into an image point at the position $\vec{q}$ with

$$\vec{q} = \vec{c} + \frac{R^2}{|\vec{p} - \vec{c}|^2}(\vec{p} - \vec{c}). \tag{1}$$

This is as easy to program and evaluate as the mirror image at a straight line. The inversion does not change the local angle of intersection between two lines. Looking only at a small region, the inverted image is similar to the original. Only its orientation and size change. Overall, inversion causes distortions and bends straight lines. These are important differences to mirror images that have the same size as the original and that leave straight lines intact.

To be more precise, the inverted images of circles and lines are circles and lines too, but usually with a different radius. Only circles and straight lines that are perpendicular to the inverting circle do not change upon inversion. You can see this in Figure 2: Inversion in the black circle maps the kaleidoscopic image disc onto itself. Points at the border change their position but stay on the border. This too happens for mirror symmetries at the two straight sides of the triangle. Thus points outside the border will forever stay outside and cannot be mapped into the basic triangle with these three reflections. You can find more details in Christersson's presentation [3].

The Mapping Procedure for Kaleidoscopes Using Triangles

I am using an iterated function to map the position of a pixel into the basic triangle of the three reflecting lines. It does not use the three mirror symmetries directly, which would be too complicated. Instead, we replace the mirror symmetries at the two straight lines by their dihedral group.

Each iteration first maps a point into the sector between the two straight lines using the method presented in the section "Rosettes". If the resulting position lies inside the circle of the arc then an inversion in the circle is done and another iteration starts. On the other hand, if the position lies outside the circle it has to be inside the triangle and we get the colour of the output pixel from the colour of the input image at this position.

An iteration step roughly moves a point by one unit cell of the pattern towards the center. Close to the border of the Poincaré disc the distance between a point and the border increases exponentially with the number of iterations. Thus most pixel positions require only a few iterations to get mapped into the basic triangle. If the iteration does not terminate after a maximum number of steps or if the pixel lies outside the image disc then its colour is set equal to a background colour. The limit for the number of iterations has only a small effect on the computing time and the smoothness of the border because of the finite size of pixels.

Kaleidoscopes with a Large Sum of Angles

If the sum of angles is larger than 180° we proceed similarly as for small sums of angles. Figure 3 shows an image resulting from the triangle (4 2 3). It has an angle of 45°, a right angle and an angle of 60° with a sum of 195°. Two straight lines make a four-fold rotational symmetry. The third side of the triangle is now an arc that bulges out away from the triangle. It acts similarly to a concave mirror and magnifies images far away from the center. We use the same iterative mapping as for kaleidoscopes with a small sum of angles except that now an inversion in the circle is done if the point lies *outside* the circle. If the point lies inside the circle it has to be inside the triangle and we can read out the colour from the input image at its position.

Note that the image covers the entire plane because two circles intersecting at right angles always have their centers outside the other circle. Thus a circle having its center at the intersection of the two straight lines cannot cross the circle of inversion at a right angle. For this reason there is no limiting circle as for kaleidoscopes with a small sum of angles.

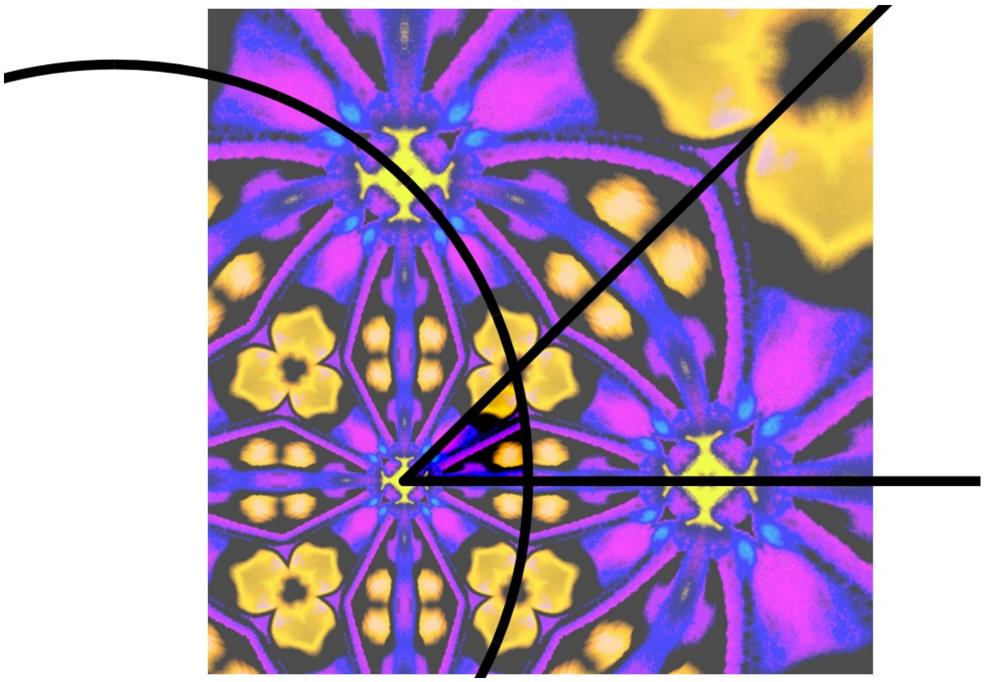

Figure 3: *Kaleidoscopic image with the 4, 3 and 2-fold rotational symmetries of an octahedron. The black lines show the reflecting sides of the basic triangle. The center of the octahedral symmetry lies near the upper left corner.*

This image is best seen as a stereographic projection of a sphere. The two straight sides of the triangle are projections of great circles on the sphere. They go through the south and north pole. The arc is a projection of a great circle on an inclined plane. Great circles are straight lines on the sphere and the triangle in Figure 3 is a projection of a spherical triangle. Its reflections create a repeating image on the sphere which has the symmetry of an octahedron. In Figure 3 we see a stereographic projection of such an image because the mirror image at a great circle corresponds to an inversion in its projected image. The projection causes large distortions and increasing sizes going away from the center. This is opposite to the projection from hyperbolic space into the Poincaré disc. For more images see my album "kaleidoscopic" [11]. You can find more about polyhedral symmetries and related images in "Creating Symmetry" [6]. Spherical kaleidoscopes create images of spheres with symmetric repeating decorations. They can be made with plane mirrors that form pyramids instead of prisms [10].

Kaleidoscopes Using Regular Polygons

A regular polygon has sides of equal length meeting at equal corner angles. Reflections at its sides create a kaleidoscopic image if the corner angle is an unit fraction of 180°. Using arcs as sides we can have regular polygons with any number k of corners generating any desired h-fold rotational symmetry at its corners. Inversions at the circles gives a tiling of hyperbolic space with these polygons [5, 7]. Each corner is shared by $2h$ polygons. Half of them are mirror images of the original polygon. Figure 4 shows the result of a regular pentagon. Its right corner angles create five different centers of two-fold rotational symmetry. Two neighboring corners together generate a frieze with mirror symmetries. Thus there are five different friezes crossing at right angles. Christersson [2] too creates such images. They are always rather asymmetric and appear to be chaotic, especially for smaller corner angles. My album "hyperbolic wallpapers" [12] shows other images of higher resolution. M. C. Escher's astounding "Circle Limit" series of prints uses tilings of the hyperbolic plane with polygons.

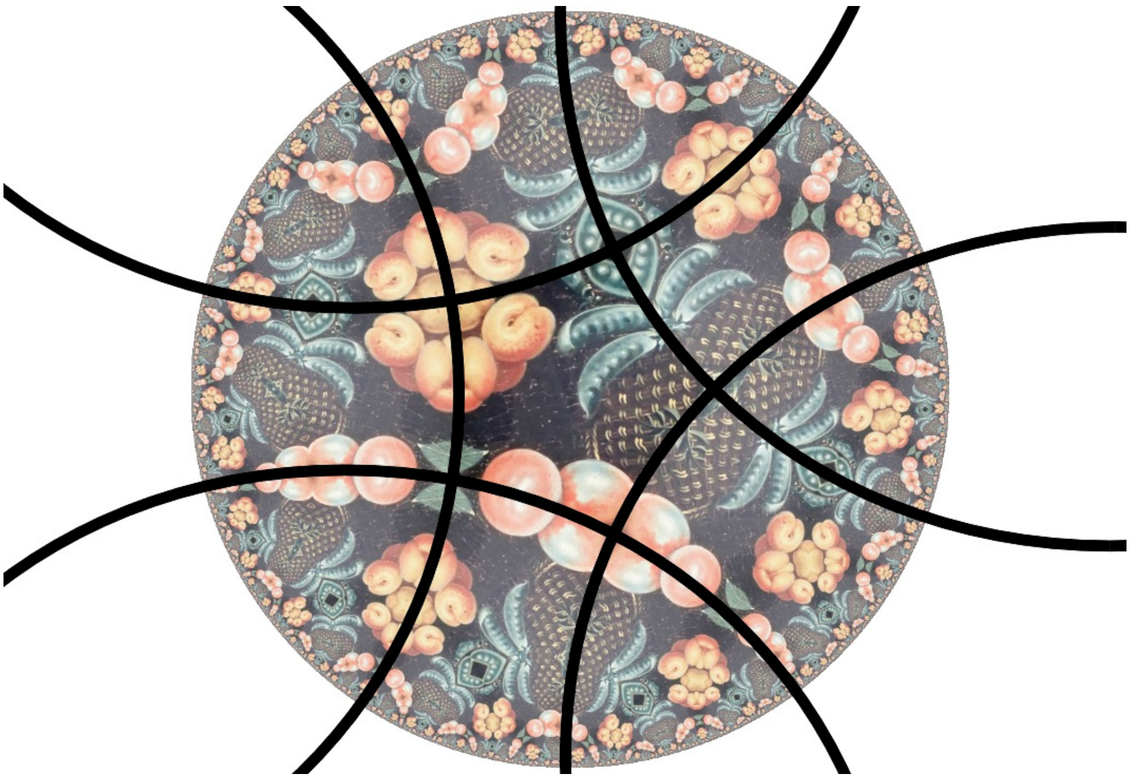

Figure 4: *Kaleidoscopic image resulting from a regular pentagon with right corner angles. The black circles show the reflecting arcs.*

The Mapping Procedure for Kaleidoscopes Using Regular Polygons

We have to map points into the polygon using reflections at its sides. The symmetries of regular polygons make this easier than for triangles. As long as a point lies outside the polygon we choose the arc that lies closest to the point. Then we make an inversion in the circle belonging to this arc. We use polar coordinates and a coordinate system with its origin at the center of the polygon. The polygon has k corners and is centered at the origin. Its corners thus lie at polar angles of $i * (360°/k)$, where i goes from 1 to k. Its sides are arcs connecting two adjacent corners and we have to find the arc that covers the polar angle ϕ of the point that we are mapping. To do this we simply get the integer number i with $i(360°/k) < \phi < (i+1)(360°/k)$, all obviously modulo 360°. This defines the arc that is closest to the point. It is part of the circle that has its center at the polar angle $(i + 1/2)(360°/k)$. If the point lies outside this circle then it is inside the polygon and we can get the colour for the output pixel from an input image at its position. If the point lies inside the circle we invert it at this circle and repeat the procedure. The resulting image is a Poincaré disc that has its center at the center of the polygon. Its border has right angles to the reflecting circles. Thus the polygon has straight sides in hyperbolic space.

Kaleidoscope Using Regular Polygons and Rotational Symmetry

To get a more pleasing image we cover the polygon at the center with a symmetric input image similarly to Christersson [2]. We could use any n-fold rotational symmetry if n is a divisor of the number k of the polygons corners. For $n = k$ a mirror symmetric rosette such as presented in the section "Rosettes" would give the same result as a kaleidoscope with a triangle (k 0 2h), where $2h$ is the number of regular polygons sharing a corner. To get new designs we use a rosette with rotational symmetry and no mirror symmetry. We create such a rosette with a mapping [6] that replaces a point with polar coordinates (r, ϕ) by a point with a multiplied angle $(r^p, k\,\phi)$. This point is then used to look up the colour in the input image. Note that rotating a point by an angle of $360°/k$ around the origin gives the same mapped point and the same colour for the resulting image. Thus it is of k-fold rotational symmetry. The exponent p for r is an arbitrary number. A reasonable choice is $p = 2$. Figure 5 is based on a regular pentagon with corner angles of 60° and a decoration of five-fold rotational symmetry. Thus this figure has cyclic symmetry. More images of high resolution are in my album [12].

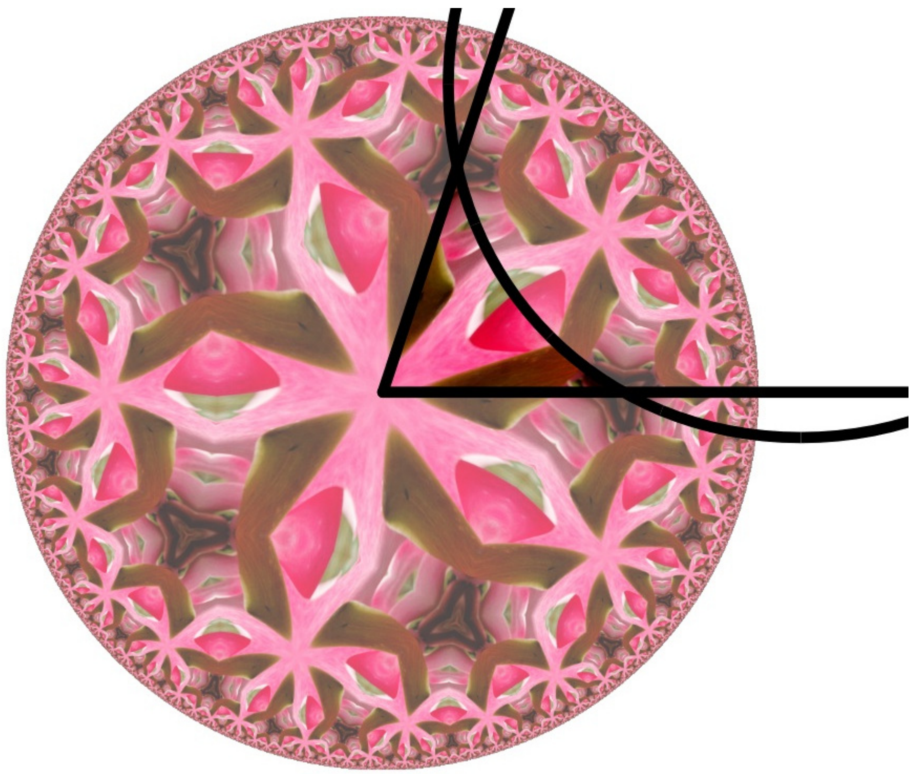

Figure 5: *Kaleidoscopic image resulting from a regular pentagon and a rosette with 5-fold rotational symmetry as input image. The black lines show the borders of the basic motif. Only the arc is reflecting.*

Summary and Conclusions

Using circle arcs instead of straight lines and inversion in circles as reflections we can create kaleidoscopes with otherwise impossible rotational symmetries. For n-fold rotational symmetries with large n and small corner angles we get Poincaré disc representations of periodic decorations on hyperbolic space. The resulting images can only be created using mathematical transformations. Larger angles result in images with planar and spherical geometry, which can both be obtained with optical mirrors too. Wallpapers for hyperbolic space with rotational symmetry are presented. They have cyclic symmetry instead of the dihedral symmetry of kaleidoscopic images. Efficient iterative methods for making these images are presented. They create a mapping between the pixels of an output image with the desired symmetries and the pixels of an arbitrary input image.

You can find more information than presented here in my blog [13], Wikipedia, Wolfram Math World and the Wolfram Demonstrations Project. Kaleidoscopic images of high resolution are in my Flickr albums [11, 12]. Farris' book "Creating Symmetry" [6] is a good resource for doing your own computer experiments and for studying symmetries. You can find more about repeating patterns on the plane, the sphere and hyperbolic space in "The Symmetry of Things" [4] and "Noneuclidean Tesselations and Their Groups" [8]. To make your own optical kaleidoscopes with real mirrors you could consult David Joyces "Kaleidoscope Mirror Designs" [10].

Acknowledgements

This work has grown out of reading Farris beautiful book "Creating Symmetry" [6]. I am grateful to Frank Farris for encouraging comments.

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