Experiencing Group Structure: Observing, Creating and Performing the Plain Hunt on 4 via Music, Poetry, Visual and Culinary Arts

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Abstract

This workshop introduces participants to a group structure, derived from change-ringing, and its interpretation using a variety of artistic modalities. The purpose of the proposed activities is to have participants experience diverse multisensory embodiments of equivalent structures, in a process that may contribute to more robust mathematical understanding. The workshop will span multiple artistic modalities including, music, poetry, visual, and culinary arts.

Introduction

Mathematical structures are born of mathematicians' experiences and observations of the world. These observations are crystalized through elegant expression, notation, operations and extensions on patterns noticed across many phenomena. Moving in the other direction, mathematics educators sometimes struggle to re-embody abstract mathematical relationships in multiple and meaningful ways, to help learners experience mathematical equivalence across diverse representations, and build robust conceptual understanding.

The workshop authors have a long-time research interest in finding ways to explore conceptual learning in mathematics education through hands-on, participatory artistic maker activities (including textile arts, sculpture, music, dance, poetry and drama), at different scales, and with different levels of collaborative engagement (from individual or pair work to large group efforts) [4, 5, 7, 8, 9, 10, 12]. From an educational perspective, experiencing widely varied multisensory embodiments of abstract relationships, and understanding their structural equivalence, contribute to a deeper and more robust mathematical understanding [1, 2, 12, 14]. Introducing aesthetically-pleasing artistic mathematical embodiments to math classes has the potential to bring pleasure to a subject that is too often treated as dry and dull.

In this workshop, we will explore the structure of the algebraic group that we will refer to as the *Plain Hunt on 4* (PH4), a subgroup of the permutations on a set of four objects. The workshop will engage participants in individual and collaborative art-making and performance in the realms of tonal and percussive music making, poetry, and visual and culinary arts to make and explore aspects of group structure.

Plain Hunt on 4

The Plain Hunt is a simple pattern from the practice of change ringing, the activity of performing mathematical permutations in the ringing of church tower bells. Change ringing dates back to early 17th century England, and is practiced worldwide today, sometimes using handbells as an alternative to tower bells. Hart has explored change ringing as an application of combinatorics [6], and Sayers' mystery novel *The Nine Tailors* uses change ringing as a key plot point [13]. Borkovitz's and Schaffer's earlier Bridges papers refer to change ringing and dance in their consideration of the full set of permutations on 4 elements [15, 16, 17]. Change ringing involves a team of people collaboratively ringing a set of tuned bells in prearranged permutation sequences. In Hart's film, for example, a team plays a sequence known as Grandsire Caters on 10 bells. As Hart points out, change ringing permutation sequences operate under the constraint that no bell can move more than one position from one line of the permutation to the next, because of the weight and inertia of the large church bells.

Figure 1: *A schematic diagram of PH4*

The Plain Hunt on 4 (PH4) is a simple traditional change-ringing sequence on four bells that swaps the positions of the first and last pairs of bells, and then of the middle pair of bells, recursively (see Figure 1). The structure of PH4, treated as a group structure generated by two elements and realized and explored in many different artistic media, will be the mathematical structure at the focus of our workshop.

Using the four tones of the well-known Westminster Chimes from the Parliament Buildings in London (A=1, G=2, F=3, C=4) as an example, Table 1 and Figure 2 give an account of PH4 starting with a descending line of pitches (known as 'rounds') and their corresponding numbers.

Table 1: *PH4 diagram using numbers 1-4 and the Westminster Chimes pitches A, G, F, C*

Step 1	1/A	2/G	3/F	4/C	Start in rounds (bells in descending pitch order)
Step 2	2/G	1/A	4/C	3/F	
Step 3	2/G	4/C	1/A	3/F	
Step 4	4/C	2/G	3/F	1/A	
Step 5	4/C	3/F	2/G	1/A	Reverse rounds (ascending pitch order)
Step 6	3/F	4/C	1/A	2/G	
Step 7	3/F	1/A	4/C	2/G	
Step 8	1/A	3/F	2/G	4/C	
(Step 9)	(1/A)	(2/G)	(3/F)	(4/C)	(Return to original rounds)

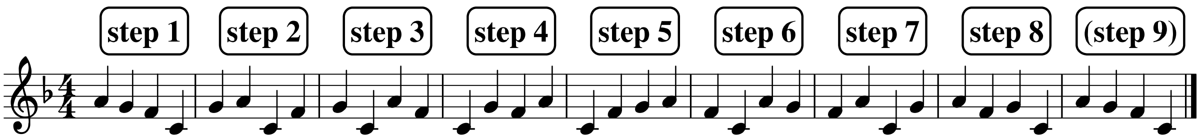

Figure 2: *PH4 on descending Westminster Chimes tones in music notation*

PH4 as a Group Structure

An algebraic group is a set of elements together with an operation $*$ acting on the objects with the conditions that the operation is closed on the set, the operation is associative, there is an identity element $\boldsymbol{e}$, and every element has an inverse. The identity element is unique, inverses are unique and denoted by $\boldsymbol{x^{-1}}$, and when the operation is understood we write $\boldsymbol{xy}$ for $\boldsymbol{x} * \boldsymbol{y}$. See [3] for more mathematical details.

Consider the group with 8 elements that is relevant for this workshop. The elements of this group are actions, and the operation is composition. We first define two elements $\boldsymbol{a}$ and $\boldsymbol{b}$ that act on strings of 4 symbols as follows: $\boldsymbol{a}$ switches the first and second and switches the third and fourth, so ex. $\mathbf{1\,2\,3\,4} \xrightarrow{\boldsymbol{a}} \mathbf{2\,1\,4\,3}$; $\boldsymbol{b}$ switches the second and third, so ex. $\mathbf{1\,2\,3\,4} \xrightarrow{\boldsymbol{b}} \mathbf{1\,3\,2\,4}$. Notice that if we apply either $\boldsymbol{a}$ or $\boldsymbol{b}$ twice in a row, we get back to where we started:

$$\mathbf{1\,2\,3\,4} \xrightarrow{\boldsymbol{a}} \mathbf{2\,1\,4\,3} \xrightarrow{\boldsymbol{a}} \mathbf{1\,2\,3\,4}, \text{ or } \mathbf{1\,2\,3\,4} \xrightarrow{\boldsymbol{b}} \mathbf{1\,3\,2\,4} \xrightarrow{\boldsymbol{b}} \mathbf{1\,2\,3\,4}$$

Also notice that the order we apply the actions matters:

$$\mathbf{1\,2\,3\,4} \xrightarrow{\boldsymbol{a}} \mathbf{2\,1\,4\,3} \xrightarrow{\boldsymbol{b}} \mathbf{2\,4\,1\,3}, \text{ versus } \mathbf{1\,2\,3\,4} \xrightarrow{\boldsymbol{b}} \mathbf{1\,3\,2\,4} \xrightarrow{\boldsymbol{a}} \mathbf{3\,1\,4\,2}$$

If we start with a string of 4 symbols and alternate between $\boldsymbol{a}$ and $\boldsymbol{b}$, we get back to the start in 8 (and no fewer) steps; ***a*** and ***b*** are their own inverses, so any even power of ***a*** or ***b*** is the identity, any odd power reduces to just ***a*** or just ***b***, and ***abababab***=the identity.

$$\mathbf{1\,2\,3\,4} \xrightarrow{\boldsymbol{a}} \mathbf{2\,1\,4\,3} \xrightarrow{\boldsymbol{b}} \mathbf{2\,4\,1\,3} \xrightarrow{\boldsymbol{a}} \mathbf{4\,2\,3\,1} \xrightarrow{\boldsymbol{b}} \mathbf{4\,3\,2\,1} \xrightarrow{\boldsymbol{a}} \mathbf{3\,4\,1\,2} \xrightarrow{\boldsymbol{b}} \mathbf{3\,1\,4\,2} \xrightarrow{\boldsymbol{a}} \mathbf{1\,3\,2\,4} \xrightarrow{\boldsymbol{b}} \mathbf{1\,2\,3\,4}$$

We recognize these permutations as the same permutations of PH4 in Table 1. There are $\mathbf{4}! = \mathbf{24}$ arrangements of 4 distinct symbols. PH4 includes 8 of these permutations. If we started with a permutation that is not included in PH4, we could still apply our actions to get 8 distinct permutations:

$$\mathbf{1\,2\,4\,3} \xrightarrow{\boldsymbol{a}} \mathbf{2\,1\,3\,4} \xrightarrow{\boldsymbol{b}} \mathbf{2\,3\,1\,4} \xrightarrow{\boldsymbol{a}} \mathbf{3\,2\,4\,1} \xrightarrow{\boldsymbol{b}} \mathbf{3\,4\,2\,1} \xrightarrow{\boldsymbol{a}} \mathbf{4\,3\,1\,2} \xrightarrow{\boldsymbol{b}} \mathbf{4\,1\,3\,2} \xrightarrow{\boldsymbol{a}} \mathbf{1\,4\,2\,3} \xrightarrow{\boldsymbol{b}} \mathbf{1\,2\,4\,3}$$

and

$$\mathbf{1\,3\,4\,2} \xrightarrow{\boldsymbol{a}} \mathbf{3\,1\,2\,4} \xrightarrow{\boldsymbol{b}} \mathbf{3\,2\,1\,4} \xrightarrow{\boldsymbol{a}} \mathbf{2\,3\,4\,1} \xrightarrow{\boldsymbol{b}} \mathbf{2\,4\,3\,1} \xrightarrow{\boldsymbol{a}} \mathbf{4\,2\,1\,3} \xrightarrow{\boldsymbol{b}} \mathbf{4\,1\,2\,3} \xrightarrow{\boldsymbol{a}} \mathbf{1\,4\,3\,2} \xrightarrow{\boldsymbol{b}} \mathbf{1\,3\,4\,2}$$

The actions can be represented by a group that is generated by the elements $\boldsymbol{a}$ and $\boldsymbol{b}$, which satisfy these three conditions:

$$\boldsymbol{a^2 = e, b^2 = e, abababab = (ab)^4 = e}$$

An effective way to list all of the elements in the group is as follows: $\{\boldsymbol{e, a, ab, aba, abab, ababa, ababab, abababa}\}$. Note that our convention is to read the actions from left to right, so $\boldsymbol{ab}$ means do $\boldsymbol{a}$ first, then do $\boldsymbol{b}$. Another way to list the elements is to simplify by using the conditions of the group to get $\{\boldsymbol{e, a, ab, aba, abab, bab, ba, b}\}$.

Because $\boldsymbol{a}$ and $\boldsymbol{b}$ are their own inverses, the arrows in the 8 steps above could all be reversed. So we could start with $\boldsymbol{b}$ instead of $\boldsymbol{a}$, but as long as we alternate between the two actions then it takes 8 steps to get back to the start. Figure 3 is the Cayley diagram for the group. The vertices are elements of the group and the edges show which operation ($\boldsymbol{a}$ or $\boldsymbol{b}$) is applied to a given element.

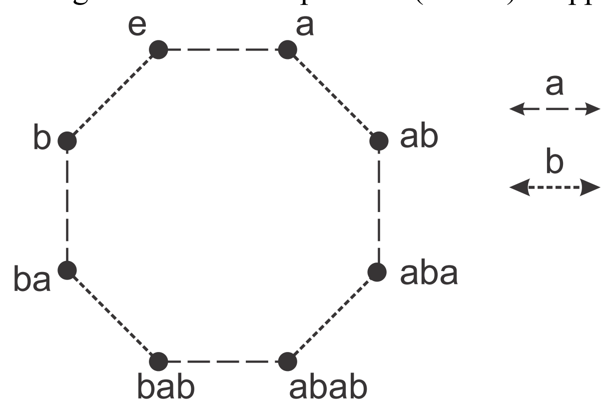

Figure 3: *Cayley Diagram for PH4*

Workshop

The workshop will give participants the opportunity to experience algebraic group structure through observing, creating and performing individual and collaborative representations of the PH4 group as melodic and percussive music, an interesting permutational poetic form, the performative co-creation of a work of visual art, and PH4 as edible culinary art.

Conceptual and historical introduction. The workshop will begin with a short talk and slides introducing the Plain Hunt on 4 in change ringing, its structure, and an analysis of PH4 as a subgroup of the permutation group on a finite set with cardinality 4.

Varieties of experiential modalities. In designing experiential activities that embody PH4, the authors realized that within each of the artistic forms, there are additional performative variables (or modalities) that affect qualitative aspects of the experience. These include distinctions between individual and collaborative work, performative/temporal and visual/spatial outcomes, and sequential and simultaneous creative activity (see Table 2).

What is more, a 'swapping' algorithm like PH4 or any other change ringing sequence can be carried out in several variants: (a) by swapping temporal spots in a performative sequence (as in traditional tower bell ringing; (b) by physically swapping bells (or sets of coloured cards, or words) while keeping one's own bodily position invariant, and then playing out the sequence in prearranged order (left-to-right, for example); (c) by physically swapping one's body position in a linear arrangement with other collaborative participants while keeping the bells (or coloured cards) invariant, and then playing out the sequence in prearranged order (left-to-right, for example).

Table 2: *Comparing aspects of experiences*

	Bell ringing	Piano	Drum kit	Card swapping	2 people plaiting	Braided bread	Poetry
Individual (I) or Collaborative (C)	C	I	I	C	C	I/C	I/C
Performative/temporal (P) or Visual/spatial (V)	P	P	P	P/V	P/V	P/V	P
Sequential (SE) or Simultaneous (SI)	SE	SE	SE	SI	SI	SE	SE

Within traditions of handbell ringing, variant (b) is termed ‘lapping’ or ‘cross and stretch’, as handbells are placed on a table in front of the bellringers, and the bells are physically swapped while the players remain stationary. Variant (c) is termed ‘body ringing’, and the players physically swap places and move up and down the line through these swaps.

We will experiment with all three of these variants as part of collaborative handbell ringing, poetic performance and card swapping visual art creation experiences, so that participants can compare the varied aspects of mathematical understanding of PH4 that may follow from each of these modalities.

Workshop Activities

1. **PH4 in tonal music on handbells and piano.** Participants will have the opportunity to explore PH4 sequences through individual and collaborative tonal music making. Workshop leaders will supply sets of 4 handbells and a portable roll-up piano synthesizer keyboard, and will lead the group in ringing the changes in a linear temporal performance through temporal, lapping and body ringing swaps with handbells, and in individual temporal performances using the linear, visual layout of the piano keyboard.
2. **PH4 in percussive music on a drum kit.** Workshop leaders will supply a drum kit or its equivalent, in which the sequencing of PH4 is performed individually using all four limbs (left and right, arms and legs) in a square, rather than a linear arrangement. It is anticipated that the swaps of large movements of the four limbs, involving an individual’s proprioception and balance [11], may be more difficult to accomplish than the bellringing and keyboard experiences, but may also have the potential for developing a deeper connection with the PH4 permutations through more full-bodied engagement.
3. **PH4 in visual art creation through card-swapping.** Participants will recreate Knoll’s *Colourwave* visual art pieces [9] (see Figure 4) through swapping activities on PH4 with decks of rectangular and circle-segment coloured cards. Workshop leaders will supply the cards and lead the activity sequences, which will leave the ‘trace’ of the static art pieces through the performance of the swaps, using lapping and body ringing techniques.

Figure 4: *Colourwave on 20, Bounce and Torus*

PH4 in poetry on strings of four words (or syllables). Workshop leaders will supply blank cards as well as some pre-made word and syllable cards, so that participants may work either with the organizers' word choices or their own choices. Working individually (in writing) and collaboratively (in performance with others), participants will have the chance to create poems through PH4 permutations on strings of four words or syllables, using temporal sequencing, lapping and body ringing variant techniques. (See Table 3 for two examples of PH4 poetry.)

Table 3: *PH4 permutational plaited poetry*

Desert Poem:			
Wings	over	dry	land
Over	wings,	land	dry
Over	land,	wings	dry
Land	over	dry	wings
Land	dry	over	wings
Dry	land	wings	over
Dry	wings	land	over
Wings	dry	over	land
(Wings	over	dry	land.)

Yoda & Friend			
You	had	me	there.
Had	you	there	me?
Had	there	you	me
There	had	me	you
There	me	had	you
Me	there	you	had
Me	you	there	had
You	me	had	there
(You	had	me	there.)

4. **PH4 in culinary arts.** Knoll will have access to baking facilities before the workshop to produce edible examples of breads braided in the PH4 pattern. Workshop leaders will supply long licorice whips that participants can plait into edible embodiments of PH4.
5. **Other examples** may include ribbon or plasticine plaits using multiples of four colours as illustrations of PH4 as a cyclic subgroup of the permutation group on a finite set of degree 4 (see Figures 5 and 6). Activities may also explore PH on numbers other than 4, and chiral variations that over/under patterns introduce in braid groups (beyond the scope of this paper).

Figure 5: *The three distinct PH4 subgroups of the 4! permutations on a set of 4 coloured ribbons*

Figure 6: *PH4 plaited in plasticine*

Presentation at informal music night. We plan to enlist interested participants to perform examples of our PH4-generated permutational poetry, set to a PH4 piano - handbell - percussion accompaniment, as well as a performance of Colourwaves visual artmaking through colour card swapping as a Bridges Informal Music Night collaborative contribution.

Closing reflections and discussion. In the closing portion of the workshop, participants will have the chance to reflect on the different qualities of learning and understanding generated by the variety of artistic, sensory, temporal/spatial, individual/collaborative and other experiential modalities they have engaged in during this exploration of PH4.

Conclusion

Mathematicians experience and observe structures in the natural world, noticing when they are equivalent across sensory modalities. Mathematics educators struggle to convey this equivalence to learners. We propose that experiencing diverse multisensory embodiments of equivalent structures can contribute to better mathematical understanding. This workshop presents an instance of such a structure and its associated understanding.

References

[1] A. Arcavi. "The Role of Visual Representations in the Learning of Mathematics." *Educational Studies in Mathematics*, vol. 52, no. 3, 2003, pp. 215–241.

[2] M.E. Brenner, R.E. Mayer, B. Moseley, T. Brar, R. Durán, B.S. Reed and D. Webb. "Learning by Understanding: The Role of Multiple Representations in Learning Algebra." *American Educational Research Journal,* vol. 34, no. 4, 1997, pp. 663–689.

[3] J.B. Fraleigh. *A First Course in Abstract Algebra*. Addison Wesley, 2002.

[4] S. Gerofsky. "Learning Mathematics Through Dance." *Bridges Conference Proceedings*, Enschede, The Netherlands, July 27–31, 2013, pp. 337–344. archive.bridgesmathart.org/2013/bridges2013-337.html.

[5] S. Gerofsky, E. Knoll and J. Forren. "Dancing Rope and Braid Into Being: Whole-body Learning in Creating Mathematical/Architectural Structures." *Bridges Conference Proceedings*, Waterloo, Canada, July 27–31, 2017, pp. 523–530. archive.bridgesmathart.org/2017/bridges2017-523.html.

[6] G. Hart. Mathematical impressions: Change ringing. Film retrieved February 2018 from www.simonsfoundation.org/2014/02/03/mathematical-impressions-change-ringing/.

[7] E. Knoll, W. Landry and T. Taylor. "Mat Weaving: Towards the Möbius Band." *Bridges Conference Proceedings*, Enschede, The Netherlands, July 27–31, 2013, pp. 579–586. archive.bridgesmathart.org/2013/bridges2013-579.html.

[8] E. Knoll, W. Landry, T. Taylor, P. Carreiro and S. Gerofsky. "The Aesthetics of Scale: Weaving Mathematical Understandings." *Bridges Conference Proceedings*, Baltimore, USA, July 29–Aug. 1, 2015, pp. 533–540. archive.bridgesmathart.org/2015/bridges2015-533.html.

[9] E. Knoll. "Colourwave: Some Variations on a Mathematical Schema." *Journal of Mathematics and the Arts*, DOI: 10.1080/17513472.2017.1323573, 2017.

[10] E. Knoll, T. Taylor, W. Landry, P. Carreiro, K. Puxley and K. Harrison. "The Aesthetics of Colour in Mathematical Diagramming." *Bridges Conference Proceedings*, Waterloo, Canada, July 27–31, 2017, pp. 563–570. archive.bridgesmathart.org/2017/bridges2017-563.html.

[11] M. Paterson. *The Senses of Touch*. Bloomsbury Academic, 2007.

[12] N. Presmeg. "Research on Visualization in Learning and Teaching Mathematics." In *Handbook of research on the psychology of education*, pp. 205–236, 2006.

[13] D.L. Sayers. *The Nine Tailors*. Hodder, 1934/2017.

[14] R.R. Skemp. "Relational Understanding and Instrumental Understanding". *Mathematics Teaching*, 77, pp. 20–26. Oxford University Press, 2004.

[15] D.K. Borkovitz. "A Temari Permutation Sampler." *Bridges Conference Proceedings*, Waterloo, Canada, July 27–31, 2017, pp. 363-366 <archive.bridgesmathart.org/2017/bridges2017-363.pdf>

[16] K. Schaffer. "Dances of Heavenly Bodies." *Bridges Conference Proceedings*, Enschede, Netherlands, July 27–31, 2013, pp. 543-546. <archive.bridgesmathart.org/2013/ bridges2013-543.pdf>

[17] D. Roaf and A. White. "Ringing the Changes: Bells and Mathematics." In J. Fauvel, R. Flood and R.Wilson, editors, *Music and Mathematics: From Pythagoras to Fractals*, pages 113-129.