Constructing Linkages for Drawing Curves

Barbara Kimeswenger[1] and Georg Grasegger[2] and Sandra Reichenberger[3]

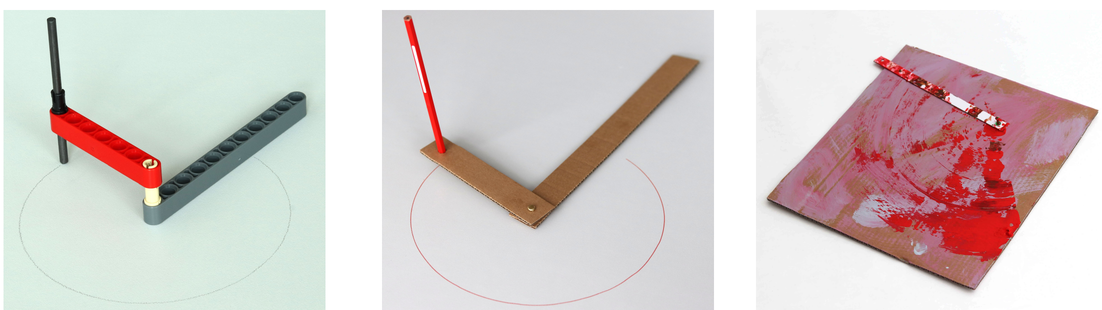
[1]Private University of Education, Diocese of Linz, Austria; barbara.kimeswenger@ph-linz.at
[2]Johann Radon Institute for Computational and Applied Mathematics (RICAM),
Austrian Academy of Sciences (ÖAW), Austria; georg.grasegger@ricam.oeaw.ac.at
[3]Johannes Kepler University Linz, Austria; sandra.reichenberger@jku.at

Abstract

In this workshop we construct mechanical devices (linkages, mechanisms) which can draw algebraic curves. Participants learn the connections of mathematics, mechanics and artistic drawings by actively constructing linkages using daily life materials.

Introduction

In order to draw a circle we know from school to use a compass or a cord with a pen. Instead we can also construct a linkage consisting of two bars connected by a rotational joint. When we fix one bar, the other one rotates and using a pen at the end of the bar we can draw (most of) a circle $x^2 + y^2 = 1$ (Figure 1).



(a) *Linkage in Lego* **(b)** *Linkage in cardboard* **(c)** *Painting with linkage*

Figure 1: *Linkages drawing a circle/disc*

In [4] linkages are constructed with Lego parts to conduct students in exploration of curves and linkages. The students physical experience is complemented by dynamic worksheets in GeoGebra. In this workshop we use daily life materials like cardboard and split pins additionally to the Lego parts. While Lego linkages are more precise the cardboard linkages are inexpensive and can be taken home which allows participants to continue their own investigations in an easy way. Instead of providing the construction plan of a linkage and exploring its motions afterwards, we focus on the inverse task: *Given a curve, build a linkage which can draw this curve*. This task enforces mathematical thinking and still allows creative explorations.

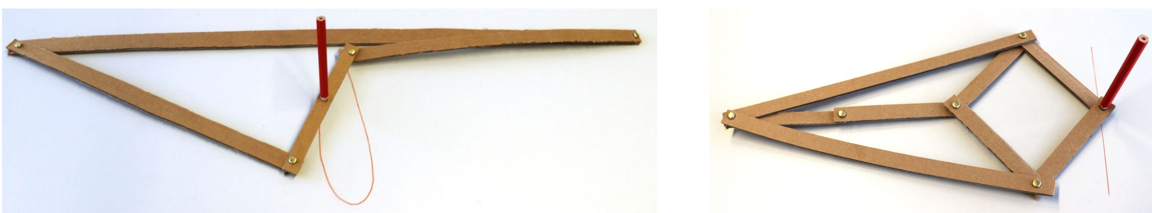
In this paper we first give an overview of the basic mathematical background, before we present the workshop description.

Linkages and Algebraic Curves

In general a linkage is a set of rigid bars (links) which are connected by rotational joints. Here we only consider planar linkages, where links only move in parallel planes. A configuration of a linkage is the description of the positions of the links at a specified moment. As in the circle linkage described above, one of the links is usually fixed. Such linkages are sometimes called mechanisms. We then explore the movements of the other links. Tracing some point of the linkage throughout its configurations yields an algebraic curve. An algebraic curve C is the zero set of a polynomial $F \in \mathbb{Q}[x, y]$, i.e.

$$C = \left\{(a, b) \in \mathbb{R}^2 \,|\, F(a, b) = 0\right\} .$$

In case of the circle we have $F(x, y) = x^2 + y^2 - 1$. In general it is not easy to construct a linkage such that the trace of some point yields a given curve. This problem has been investigated for a long time. Linkages such as those discovered by Watt or Peaucellier and Lipkin (Figure 2, compare [3]) emerged from the problem of turning rotational motions to (approximately) straight line motions.

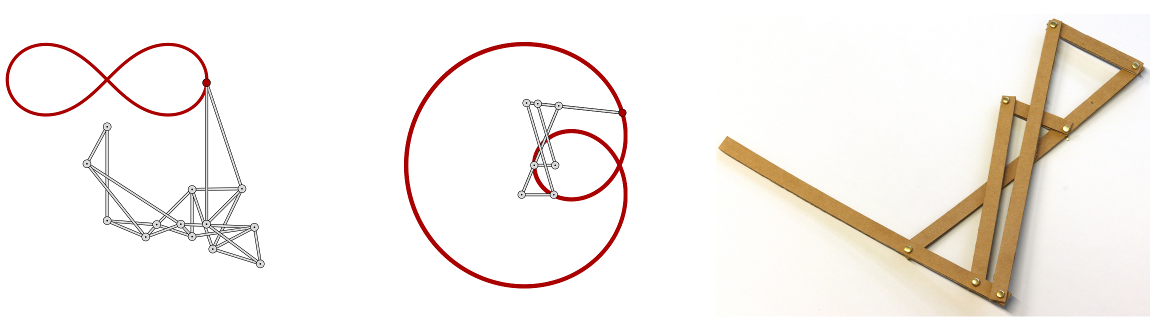


(a) *Watt linkages* **(b)** *Peaucellier-Lipkin linkage*

Figure 2: *Linkages drawing (approximate) straight lines*

Kempe [2] proved that given an algebraic curve it is possible to construct a linkage which can draw this curve. These linkages, however, are rather complicated to construct.

By simple explorations of linkages with few bars we can find some interesting cases. One of them is a linkage for drawing the figure in shape of the infinity symbol. A possible equation for such a kind of drawing is $F(x, y) = -2(x^2 - y^2) + (x^2 + y^2)^2$. This curve is known as the lemniscate of Bernoulli and it is a special case of a Cassini curve. Using a known algorithm we can find a linkage drawing this curve (Figure 3a). However, there is a much easier construction with just four links, as we explore in the workshop (Figure 4). Another curve for which we can construct a rather easy linkage is the limacon of Pascal with the general equation $-b^2(x^2 + y^2) + (-ax + x^2 + y^2)^2$. For some $a > b$ we get the linkage from Figure 3b.



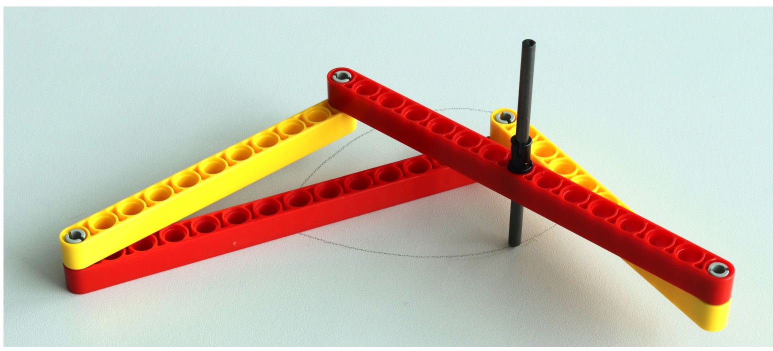
(a) *Lemniscate of Bernoulli* **(b)** *Limacon of Pascal* **(c)** *Limacon linkage in cardboard*

Figure 3: *Linkages drawing the lemniscate of Bernoulli and the limacon of Pascal*

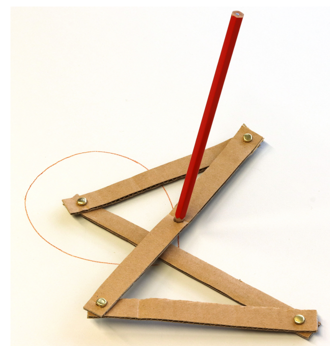
Workshop Description

In the workshop we start with linkages from this paper, following the difficulty from the easy circle to the four bar linkage with all its possible curves. We provide the curves and the daily life materials; the participants construct and explore the linkages which draw the given curves.

1. For warming up the participants are given the drawing of a circle and are asked to construct a linkage following the circle motion (as in Figure 1).
2. We give a brief introduction on the connection of linkages and mathematics (as described in the section above).
3. The participants explore the curves that can be drawn by a four bar linkage (see Figure 4). In fact we only provide the drawings of the curves from Figure 5 and might give the hint that four bars suffice. The participants need to find the correct connection of the bars, their length and the position of the pen.



(a) *Linkage in Lego*



(b) *Linkage in cardboard*

Figure 4: *Linkages drawing a figure eight. We fix the bar at the bottom level and put the pen exactly in the middle of the link in the upper most level.*

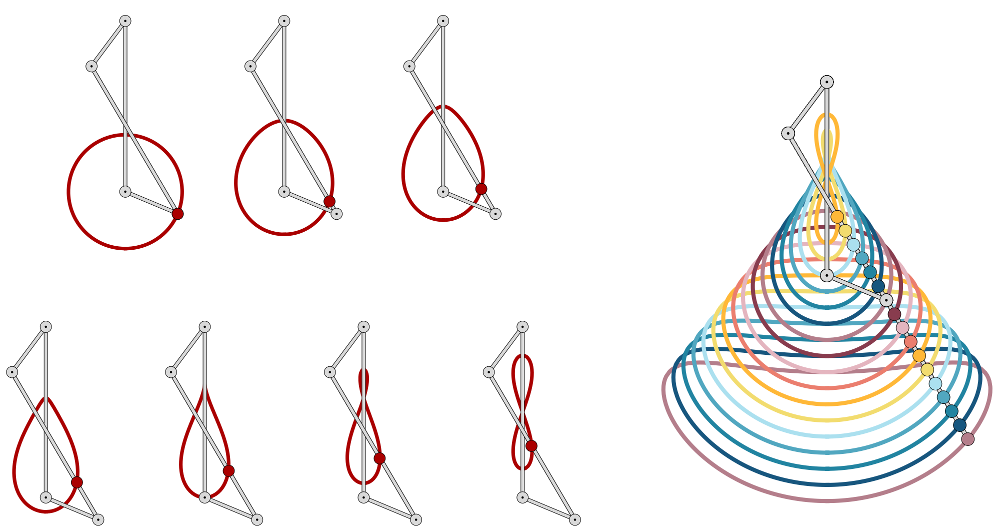


Figure 5: *Four bar linkage with different pen positions (the vertical link is fixed)*

4. Finally participants can construct more difficult linkages and create artwork by tracing not just single points but entire bars (see Figure 6).

Figure 6: *Linkage paintings*

Further Exploring

A recently published algorithm [1] allows for the construction of linkages following the motion of a rational algebraic curve. These are curves which allow a rational parametrization, i.e. a pair of rational functions $(p_1(t), p_2(t))$ such that $F(p_1(t), p_2(t)) = 0$ for almost all t. The algorithm is implemented and is available for download at www.koutschan.de/data/link. This algorithm and software were used to create the computer drawings in this paper (Figures 3a, 3b, 5).

Acknowledgments

We thank Zoltán Kovács for sharing his ideas on workshops with linkages.

References

[1] M. Gallet, C. Koutschan, Z. Li, G. Regensburger, J. Schicho, and N. Villamizar. “Planar Linkages Following a Prescribed Motion.” *Mathematics of Computation*, vol. 86, 2017, pp. 473–506.

[2] A. B. Kempe. “On a General Method of Describing Plane Curves of the nth Degree by Linkwork.” *Proceedings of the London Mathematical Society*, vol. 1-7, no. 1, 1875, pp. 213–216.

[3] A. B. Kempe. *How to Draw a Straight Line; A Lecture on Linkages.* McMillan, London, 1977.

[4] Z. Kovács and B. Kovács. “A Compilation of LEGO Technic Parts to Support Learning Experiments on Linkages.” Technical Report 1712.00440, arXiv, 2017.