Adopt a Polyhedron – A Citizen Art Project in Mathematics

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Abstract

In our science communication project *Adopt a Polyhedron* we aim to involve a general audience to consecutively realize all combinatorial types of convex polyhedra in a collaborative effort. The participants adopt a polyhedron, give it a name and make a model of it. Thus, one by one the abstract geometrical description of all polyhedra becomes concrete. Similar to the concept of Citizen Science we call this collaborative process *Citizen Art*. In the workshop each participant receives their own unique polyhedron in the form of a number and a paper template. It can be formally adopted on our website. Then the participants make an individual model of the polyhedron from a material of their choice. We bring a variety of materials. The finished models will be exhibited in the venue and pictures can be uploaded to the website in order to prove that another polyhedron has found its way into being realized. The website is available under *poly.mathematik.de*.

Introduction

The workshop is embedded in the science communication project *Adopt a Polyhedron* that is associated with the Collaborative Research Center *Discretization in Geometry and Dynamics* and aims to raise the public awareness of mathematics, with a special focus on discrete geometry. The main target groups are pupils, teachers and the general public. The workshop, as it is conducted at the conference, can also be done in schools. We provide didactical and background materials about polyhedra for teachers.

(a) **(b)**

Figure 1: *The two combinatorial types of convex polyhedra with 5 vertices: (a) a pyramid over a quadrilateral, and (b) a bipyramid over a triangle.*

Polyhedra are classical geometrical objects. In three-dimensional space the simplest one is the tetrahedron. Four non-coplanar points are always the vertices of a tetrahedron. When we move the points around (and do not end up putting them all on a plane) we will always get a geometrical shape that is some pyramid over a triangle. It can be big, small, pointed or flat, but the structure never changes. This structure, the composition of its vertices, edges, and faces is called the *combinatorial type* of a polyhedron. The combinatorial type of the body in Figure 1a is a pyramid over a quadrilateral. The quadrilateral is contained in a plane. When we lift one of the vertices of the quadrilateral above that plane, the quadrilateral breaks into two triangles and a new edge is formed. Hence the combinatorial type has changed. Table 1 shows the number of combinatorial types of polyhedra with n vertices [1].

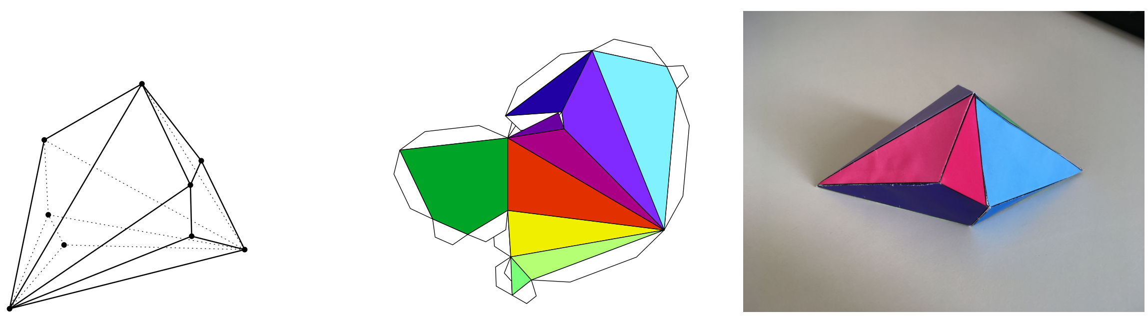
The cube ist the most prominent three-dimensional polyhedron. If we consider every die a cube, despite its often rounded edges, it is safe so say that the number of cubes on our planet ranges in the billions. Every

Table 1: *Number of combinatorial types of three-dimensional polyhedra with n number of vertices.*

number of vertices	4	5	6	7	8	9	10	11	...
comb. types of 3-dim. polyhedra	1	2	7	34	257	2606	32300	440564	...

cuboid, i.e. rectangular box with eight vertices, twelve edges and six quadrangular facets, is a cube in a combinatorial sense. By squeezing and pulling its edges, it can be morphed into a cube without adding or losing any vertices, edges, or facets. If we only look at convex polyhedra with eight vertices, there exist 256 types besides the cube. We know their combinatorial structure, but we doubt that they have all ever actually been realized in a physical way.

The main idea of our project is to involve a general audience to consecutively realize all combinatorial types of convex polyhedra in a collaborative effort. Of course this is not possible, since their number is infinite. But we can work our way up by the number of vertices. For this purpose we are setting up a website where all combinatorial types of polyhedra, sorted by their number of vertices, are published. When all combinatorial types of a certain number of vertices have been adopted, we will publish the next group. Participants of the project can sign up on our website and choose a polyhedron they wish to adopt. They can name it and download a paper model template with the unfolding of the polyhedron (Figure 2b). We also provide data for 3D printing. After assembling the paper model the participants can chose a material and make a creative model of their polyhedron. In order to *prove* that another combinatorial type of polyhedron has been realized, a photo of the model is uploaded to the website.



(a) **(b)** **(c)**

Figure 2: *(a) Computer graphic of a polyhedron with nine vertices, (b) a colorful template of a polyhedron, and (c) the assembled paper model.*

The creation of mathematical and geometrical models was – and still is – a major part of education and research (cf. von Dyck [11], Fischer [6]). In the process of making an individual model, mathematical concepts come to life. We provide materials but no detailed instructions for making the individual model. We encourage the participants to explore the object in an individual way. The level of mathematical education, as well as the dexterity and the choice of material will lead to a great variety in approaches to the problem. By diving right into the process of making a model, each participant can come up with their own process of perception. The workshop instructors are ready to help with questions about mathematics and the materials.

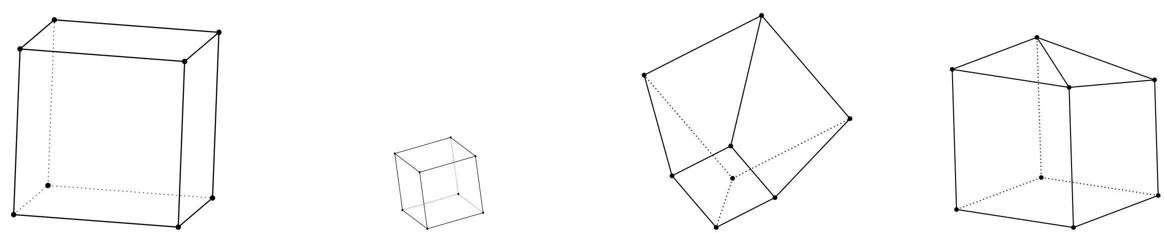
Mathematics

Convex polyhedra are classical geometrical objects that have been studied since antiquity. The classification of the platonic solids was presented in Euclid's *Elements*. About 500 years ago the painter Albrecht Dürer described and studied polyhedra in his book "A painter's manual" [5].

A convex polyhedron consists of vertices, straight edges, and flat polygonal faces. It does not have any indentations or cavities and all its inner angles between adjacent edges or faces are smaller than 180° [4].

For comparing polyhedra with each other the notion of *combinatorial equivalence* is helpful. Two polyhedra are combinatorially equivalent if their graphs have the same structure. This means that their vertices can be numbered in a way such that if two vertices of one polyhedron are joined by an edge, so are the corresponding vertices of the other polyhedron [12].

A cube consists of eight vertices, twelve edges, and six square faces, that are necessarily all of the same size. In each vertex, three edges and thus three faces meet (Figure 3a). Rotating the cube and altering its size does not change its geometrical type (Figure 3b). Looking at the polyhedron in Figure 3c, we notice that the incidences of the vertices, edges, and faces has not changed and the faces still are quadrilaterals. The combinatorial structure is the same as the cube's. If, for example, one of the quadrilaterals in broken into two triangles, a new edge emerges and we have two vertices where now four edges and thus faces meet. The incidence and therefore the combinatorial type has changed (Figure 3d).



(a) **(b)** **(c)** **(d)**

Figure 3: *(a) A cube, (b) a cube geometrically equivalent to (a), (c) a combinatorial cube and (d) a polyhedron with eight vertices that is neither a geometric nor a combinatorial cube.*

The enumeration of polyhedra uses Steinitz' theorem. It proves the isomorphism between simple, planar and three-connected graphs and the combinatorial equivalence classes of polyhedra [9]. Thus, in order to know the number and structure of polyhedra with n vertices, we need to know the number and structure of simple, planar and three-connected graphs with n vertices.

Simple, planar and three-connected graphs can be enumerated very efficiently using the program *plantri* by Brinkmann and McKay [3]. Since there are infinitely many ways to geometrically realize a polyhedron of a certain combinatorial type, we choose the Koebe–Andreev–Thurston realization in order to have a unique representation for each combinatorial type. This realization is unique up to Möbius transformation and it has a *roundish* shape, since all its edges are tangent to a sphere. Bobenko and Springborn introduced a construction technique using variational principles [2]. We followed their method to generate the polyhedra [13, Lecture 1]. In a last step we generate the nets using the free open–source mathematics software system *Sage* [10].

Plan for the Workshop

We want to encourage the participants to explore their polyhedron in an individual way. Instructors will assist the participants to find answers to mathematical questions. Additionally some posters with definitions and concepts will be on display at the venue.

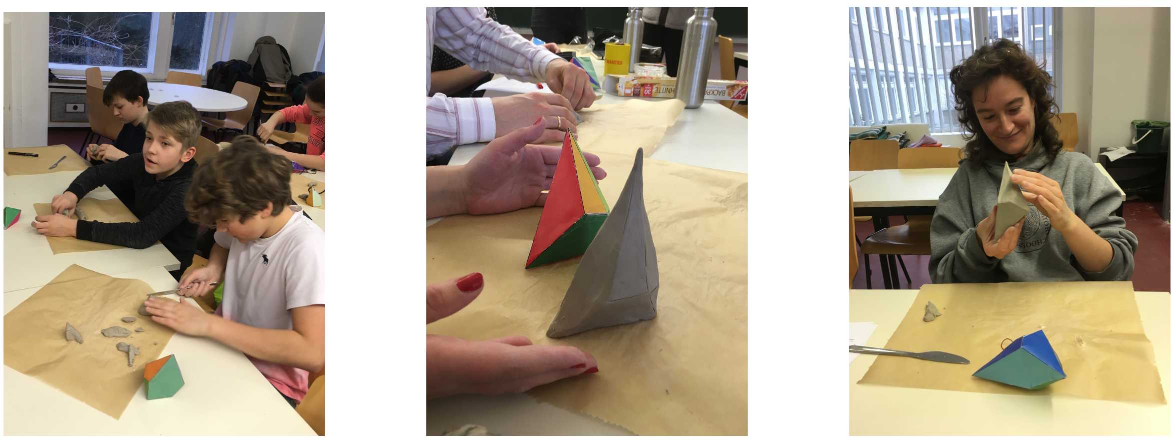
Phase 1: Adoption and Assembly

The workshop starts with the distribution of the polyhedra in form of a paper model sheet. Everyone receives a sheet and a corresponding number. The participants can log in on our website and formally adopt the polyhedron and give it a name. We will bring devices for those who do not have instant access to the internet.

The participants will cut out and glue the template in order to make a three-dimensional paper model of the polyhedron.

Phase 2: Making an Individual Model

The second phase is dedicated to building an individual model. We provide different materials from which the participants can choose. It can vary in size and geometric realization. One can recreate an unfolding out of different material and alter it in size. Another example for a model is building the skeleton of the polyhedron with straws or skewers. The skeleton is the scaffolding of the polyhedron, which consists of its vertices and edges. Creating the solid body of the polyhedron can be approached by using clay or putty. Clay requires a bit of knowledge in handling the material, putty is a softer alternative for children. Both materials allow for mistakes during the process of creating the structure. Trying to cut the polyhedron out of a block of foam is ambitious, since errors are irreversible and gauging the cutting angles by eye is challenging.



(a) **(b)** **(c)**

Figure 4: *Impressions from a similar workshop at Freie Universität Berlin.*

Phase 3: 'Picture Proof' and Presentation

The last phase brings the individual models together into the collective process of realizing all kinds of combinatorial types of convex polyhedra. Locally for the workshop, the paper models and individual models will be put on display. In order to prove that a polyhedron has found its way into realization a photo can be taken and uploaded on our website.

Didactical Background and Materials for the Classroom

The didactical concept behind the project is project-oriented learning. It focusses on finding a new and creative solution to a problem in a certain time frame [8]. We provide background material for teachers on our website. Amongst ideas and material on how to conduct the workshop in the classroom there is a general introduction on polyhedra for teachers. In Germany the conference of Ministers of Cultural Affairs defined a competence model for the mathematics curriculum [7]. The main competence as defined in this framework is *mathematical problem-solving*, since the focal point of the workshop is building a model. This activity presents the students with a problem they have to solve creatively, using mathematical thinking. They cannot rely on established calculative or mechanical schemes. The major content-related competence is the *perception of form and space*. Secondary competences could be *measuring and magnitudes* as well as *functions and equalities*, depending on the focus chosen by the teacher.

Table 2: *Linkage points of the project to mathematical curriculum for various age-groups.*

age-group	linkage point	mathematical competence
10 – 12	Polygons, angles and length. Usage of protractor.	Measuring of length and angles. Enlarging the polygons by factorization. Drawing polygons.
12 – 14	Linear functions.	Constructing the polygonal faces via intersection of linear functions.
14 – 16	Three dimensional bodies.	Calculation of area and volume of three-dimensional bodies.
16 – 18	Three dimensional coordinate system.	Construction and reading off data in a three dimensional coordinate system. 3D Data of polyhedra is an hands-on example.
16 – 18	Analytic geometry.	Considering the edges and vertices as intersections. Calculation of intersections of planes or lines in space.

Table 3: *Content of the exploration-cards.*

property	explanation
f-vector	A vector that collects the number of vertices, edges and faces of the polyhedron
Euler-characteristc	The alternating sum of the number of vertices, edges and faces is an invariant for convex polyhedra. It is always 2.
Eulerian-path	Is it possible to walk every edge of the polyhedron without walking one twice? If and only if the degree of all vertices is odd, an Eulerian path exist. The degree of a vertex is the number of joined edges.
graph of a polyhedron	The graph of a polyhedron is the union of its vertices and edges.

Teachers can present polyhedra to different age groups using age appropriate mathematical angles. In order to present the project *Adopt a Polyhedron* in the mathematics class in a meaningful way, we provide possible linkage points to the curriculum in Table 2. This is coarsely depending on the mathematics curriculum in the federal state of Berlin. We hope that international teachers can adapt this to their requirements. We have found that clay is a cheap and very interesting material for building polyhedra. Since it is relatively firm, putty is a better alternative for smaller hands. Other interesting materials are kebab-skewers with putty-joints, slim PVC pipes and wire or floral foam.

For a theoretical approach to teaching polyhedra we provide so-called *exploration cards*. They introduce features and properties of polyhedra and pose questions for the students to thoroughly examine their polyhedra. With the help of the cards, learning stations can be set up in the classroom and the students produce a fact sheet for their polyhedron. Table 3 gives an overview of the properties on the exploration cards.

Conclusion

In the workshop each participant gets to explore a unique polyhedron by making a model of it. They can adopt it on our website. These two activities provide access to geometry that directly starts with action. The idea of collectively *freeing* the polyhedra from their habitat in the sphere of abstraction, where they are known but never actually take shape is what we call *Citizen Art*. This workshop can also be done in schools and we provide background material on our website: *poly.mathematik.de*.

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