

Desargues Configuration as a Gnomonic Projection

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Abstract

This paper describes with illustrations how the expanded version of the four-dimensional solid called pentachoron can be projected to our three-dimensional space to obtain a configuration embodying the celebrated Desargues' theorem.

Introduction

In their 1936 survey into the art of mathematics, Hilbert and Cohn-Vossen described how a four-dimensional polytope called the 24-cell – when projected from its center into our three-dimensional space, yields an elegant figure called the Reye configuration [5]. The remarkable relation between four-dimensional structures and three-dimensional projective configurations is further strengthened by the fact that the gnomonic projection of the expanded pentachoron yields the Desargues configuration. This insight has previously, to the best of my knowledge, gone unnoticed.

The Gnomonic Projection of the Expanded Tetrahedron

Before constructing the actual Desargues configuration, let us examine the situation one dimension lower to set up some vocabulary and facilitate the higher-dimensional intuitions needed further on. We will achieve this by expanding the tetrahedron, and projecting it down onto a plane.

The tetrahedron is a regular solid composed of four vertices, six edges, and four triangular faces (Figure 1a). When it is subjected to the operation of *expansion* [2], its four faces move radially outwards. Its four vertices yield additional triangles, and the six edges yield squares. Thus the expanded tetrahedron, also called *the cuboctahedron* has 12 vertices, 24 edges, and 14 faces (Figure 1b). A set of edges of the cuboctahedron lying on a plane through its center constitutes a cross-sectional hexagon (an expanded triangle), and there are four such cross-sections.

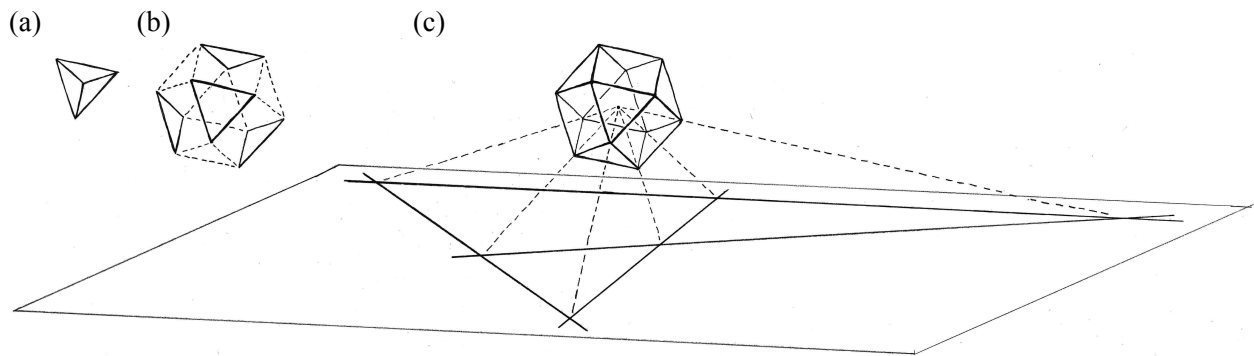


Figure 1: (a) the tetrahedron, (b) the expanded tetrahedron i.e. the cuboctahedron, (c) gnomonic projection of the cuboctahedron.

We project the cuboctahedron from its center to a two-dimensional picture plane (Figure 1c). Such projection looking at a solid from its center is called a *gnomonic projection*. The gnomonic projection

produces a depiction of the half of the structure facing towards the picture plane. Note also that a set of three edges sitting on the same plane through the center of the cuboctahedron gets projected to a single straight line in the picture plane.

The gnomonic projection of the cuboctahedron results in a system of points and lines, which in the context of projective geometry is called a *configuration* [3]. In general, this configuration is composed of six points and four lines, although depending on the direction of the view, some elements might get projected at infinity. Each of the four lines is a gnomonic projection of a cross-sectional hexagon, and the points are realized as all possible combinations of two items from the collection of these four lines.

The Desargues Configuration

To arrive at the Desargues configuration, we start with the pentachoron – a regular, four-dimensional solid composed of five vertices, ten edges, ten triangular faces, and five tetrahedral cells (Figure 2a). When it is expanded, its cells move radially outwards. Its five vertices yield additional tetrahedra, and the ten edges and the ten faces both yield triangular prisms. Thus the expanded pentachoron, also called the *runcinated pentachoron* (Figure 2b) has 20 vertices, 60 edges, 70 faces, and 30 cells. A set of faces of the expanded pentachoron lying on a 3-dimensional slice through its center constitutes a cross-sectional cuboctahedron, and there are five such cross-sections.

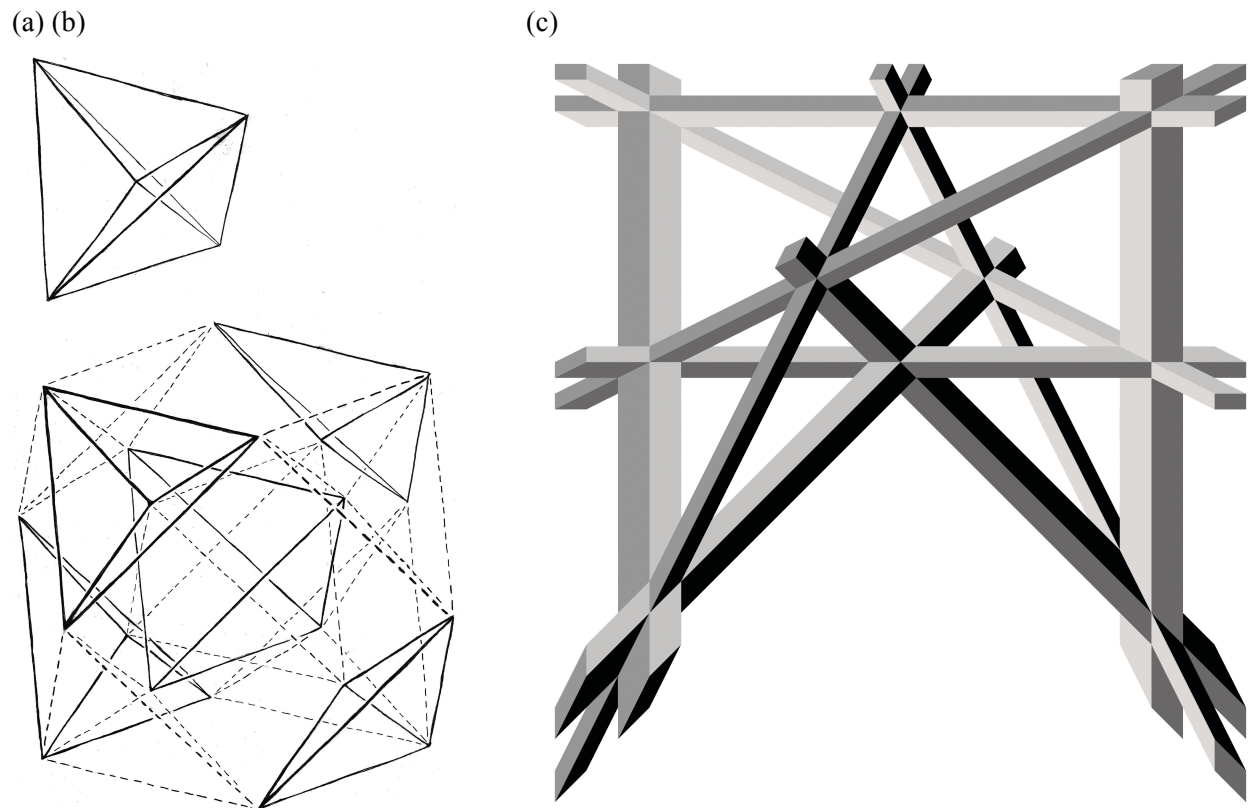


Figure 2: (a) the pentachoron, (b) the expanded pentachoron, (c) the Desargues configuration.

The gnomonic projection of the expanded pentachoron generally results in a 3-dimensional configuration composed of ten points, ten lines, and five planes. Figure 2c gives a stylized illustration of this configuration, where each plane is portrayed with a different color. From this illustration it is easy to see that each point is an intersection of three planes, and each line is an intersection of two planes. Each plane of the configuration is a gnomonic projection of a cross-sectional cuboctahedron, and the lines and the

points are realized as all possible combinations of two and three items, respectively, from the collection of these five planes.

This system of points, lines, and planes is called the Desargues configuration, and it embodies the Desargues's theorem. The theorem states that if the vertices of two triangles correspond to each other in such a way that the lines joining the corresponding vertices meet at a point, then the three meets of the extensions of the corresponding edges meet along a line, and vice versa [4]. Such relation is called a *perspectivity* – if two triangles are *in perspective from a point* (point of perspectivity), they are also *in perspective from a line* (axis of perspectivity). Although some pairs of mutually perspective triangles are more difficult to locate from Figure 2c, every point and every line of the configuration will act as a center and an axis of perspectivity, respectively, to a pair of triangles.

Generalizing to Other Dimensions

Now that we have understood the Desargues configuration as a gnomonic projection of an expanded pentachoron, we can project corresponding structures of other dimensions to produce generalizations of the Desargues' theorem. On the lower dimension we discussed above, the gnomonic projection of the cuboctahedron produced in Figure 1c embodies the fact that if two line segments are in perspective with respect to a point, they are also in perspective with respect to another point. To generalize to the higher dimension, we will project the expanded version of the 5-dimensional simplex.

The 5-simplex is a regular, five-dimensional solid composed of six vertices, fifteen edges, twenty triangular faces, fifteen tetrahedral cells, and six pentachoral hypercells. When it is expanded, its hypercells move radially outwards. Its six vertices yield additional pentachora (Figure 3a), the fifteen edges and the fifteen cells both yield tetrahedral prisms¹ (Figure 3b), and the twenty faces yield 3–3 duoprisms² (Figure 3c). Thus the expanded 5-simplex has 30 vertices, 120 edges, 210 faces, 180 cells, and 62 hypercells. A set of cells of the expanded 5-simplex lying on a 4-dimensional slice through its center constitutes a cross-sectional expanded pentachoron, and there are six such cross-sections.

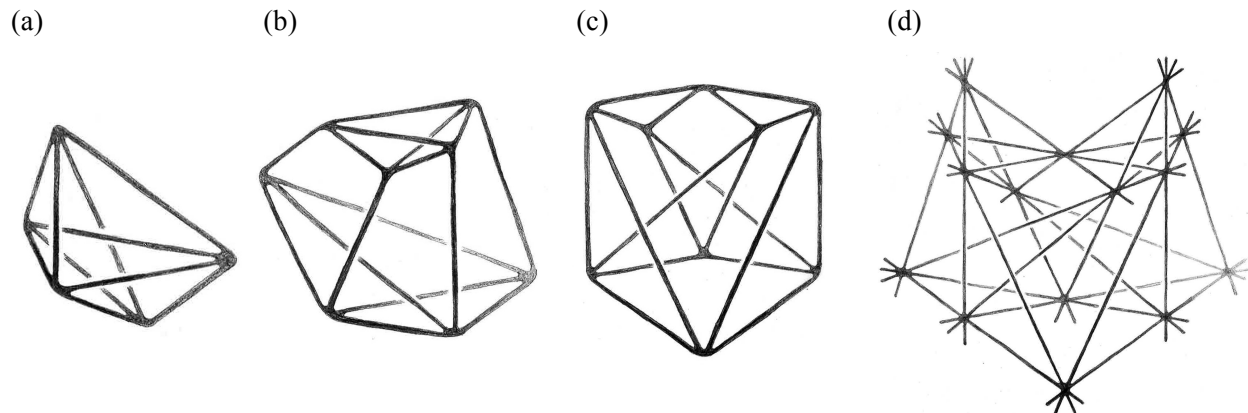


Figure 3: (a) the pentachoron, (b) the tetrahedral prism, (c) the 3–3 Duoprism, (d) the 'hyperspatial' Desargues configuration.

The gnomonic projection of the expanded 5-simplex results in a 4-dimensional configuration (Figure 3d) composed of fifteen points, twenty lines, fifteen planes and six 3-spaces, where each point is an intersection of four lines, six planes, and four 3-spaces; each line goes through three points and is an intersection of three planes and three 3-spaces; each plane goes through six points and four lines, and is an intersection of two 3-spaces; and each 3-space goes through ten points, ten lines, and five planes. Each

¹ The tetrahedral prism consists of two tetrahedra and four triangular prisms; the triangular prisms

² The 3–3 duoprism consists of six triangular prisms, coming in two linked cycles of three prisms.

3-space is a Desargues configuration, and the planes, the lines, and the points are realized as all possible combinations of two, three, and four items, respectively, from the collection of these six 3-spaces.

This configuration embodies a generalization of the Desargues theorem stating that if two tetrahedra are in perspective with respect to a point, they are also in perspective with respect to *a plane*, and vice versa. Although such generalizations of the Desargues' theorem for the perspectivity of two simplices have been described before [1], the configuration of Figure 3d affords us also with another insight – we see that if *three* triangles are in perspective with respect to a line (the meets of the extensions of the corresponding edges being collinear), they are also in perspective with respect to *another line* (the three points of perspectivity constructed from all three pairs of triangles being collinear).

In Conclusion

Although not possible to elaborate on here, it is worth mentioning that the gnomonic projection of the expanded 6-simplex results in a 5-dimensional configuration embodying the fact that if two pentachora are in perspective with respect to a point, they are also in perspective with respect to a 3-space, and vice versa. More surprisingly, it also demonstrates that if three tetrahedra are in perspective with respect to a plane (the six meets of the extensions of the corresponding edges being coplanar), they are also in perspective with respect to a line (the three points of perspectivity constructed from all three pairs of tetrahedra being collinear), and vice versa.

It seems the connection between higher-dimensional structures and projective configurations can still be the subject of interesting visual exercises in spatial imagination, and envisioning these figures is a challenging activity to an artist and a mathematician alike. Although the illustrations presented here do not depict any particular projection, and the layout of the configurations is composed purely by an artistic license, the incidence structures relevant to the stated theorems are preserved.

Acknowledgements

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