Polyhedral Models of the Projective Plane

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Abstract

The tetrahemihexahedron is the only uniform polyhedron that is topologically equivalent to the projective plane. Many other polyhedra having the same topology can be constructed by relaxing the conditions, for example those with faces that are regular polygons but not transitive on the vertices, those with planar faces that are not regular, those with faces that are congruent but non-planar, and so on. Various techniques for generating physical realisations of such polyhedra are discussed, and several examples of different types described. Artists such as Max Bill have explored non-orientable surfaces, in particular the Möbius strip, and Carlo Séquin has considered what is possible with some models of the projective plane. The models described here extend the range of possibilities.

The Tetrahemihexahedron

A two-dimensional projective geometry is characterised by the pair of axioms: any two distinct points determine a unique line; any two distinct lines determine a unique point. There are projective geometries with a finite number of points/lines but more usually the projective plane is considered as an ordinary Euclidean plane plus a line “at infinity”. By the second axiom any line intersects the line “at infinity” in a single point, so a model of the projective plane can be constructed by taking a topological disc (it is convenient to think of it as a hemisphere) and identifying opposite points. The first known analytical surface matching this construction was discovered by Jakob Steiner in 1844 when he was in Rome, and it is known as the Steiner Roman surface. A polyhedron corresponding with this surface was described by Hilbert [3], who called it simply a heptahedron (since it has seven faces, three squares and four triangles). It is a uniform polyhedron so it is included by Wenninger [6], where it is called the tetrahemihexahedron. It can be derived from an octahedron by removing alternate faces and adding the diametral squares.

The arrangement of vertices, edges and faces of the tetrahemihexahedron can be derived from the cuboctahedron (Figure 1) by identifying opposite vertices (with the corresponding edges and faces). If the cuboctahedron is taken as a tiling of the sphere, the tetrahemihexahedron is a tiling of the hemisphere with opposite points on the bounding circle identified, i.e. it is a tiling of the projective plane, so it could also be called a hemi-cuboctahedron. Identifying the opposite vertices of a polyhedron will always produce a tiling of the projective plane, although finding a corresponding physical realisation of the *abstract polyhedron* generated like this is not always easy, and the method works only with polyhedra that have inversion symmetry (reflection in the centre point does not change the polyhedron) so that the opposite elements correspond.

One consequence of the identification of opposite elements is that the resulting polyhedron has lost inversive symmetry, and the cubic symmetry of the cuboctahedron has been reduced to tetrahedral symmetry in the tetrahemihexahedron.

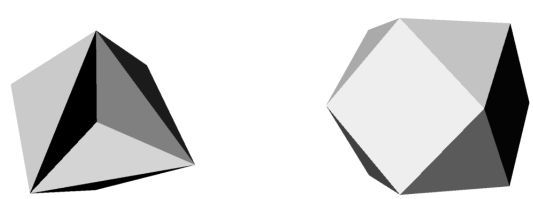

Figure 1: *The tetrahemihexahedron compared with the cuboctahedron.*

The tetrahemihexahedron is a special type of cuploid [4]. The cuploids, and their duals, all have the topology of the projective plane.

Truncation

There are several ways to create new polyhedra from existing ones. For example a realisation of the hemi-icosahedron is easily made from the tetrahemihexahedron by halving the square faces along their diagonals, recalling the observation that the regular icosahedron appears in the jitterbug transformation of the cuboctahedron into the rhombicuboctahedron. Since there are two squares at every vertex this can be done in two ways to produce different polyhedra that are mirror images of each other. Of course the faces are not all regular: six of them are isosceles right-angled triangles.

Truncation is one of the more obvious methods to use. Applying it to the tetrahemihexahedron produces the hemi-rhombicuboctahedron, just as the regular rhombicuboctahedron can be formed by truncating the cuboctahedron and adjusting the rectangular faces that are produced to become squares. The corresponding faces in the hemi-polyhedron are crossed quadrilaterals, and no such adjustment is possible, although it is possible to uncross two of the three parallel pairs. The resulting polyhedron retains the square diametral faces but in the most symmetrical version the triangles are coplanar with them, and the uncrossed quadrilaterals become trapezoidal (Figure 2).

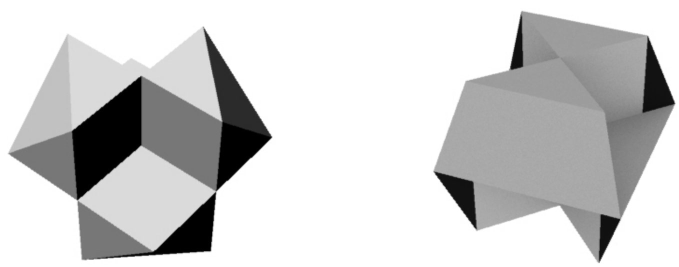

Figure 2: *The truncated tetrahemihexahedron or hemi-rhombicuboctahedron and a different realisation of the same polyhedron with some faces coplanar (the triangles are a darker shade).*

Non-planar Faces

If faces are allowed to be non-planar, hemi forms of many more polyhedra become possible. Triangles must be planar, so there are no new deltahedra, but the other Platonic polyhedra work. The hemi-cube is particularly simple with three quadrilateral faces with edges matching those of the tetrahedron (Figure 3).

The hemi-dodecahedron is particularly interesting. Its edges are those of the Petersen graph, the usual representation of which suggests an obvious way to realise it (Figure 4). One face (the base) is a regular planar pentagon. The other five each share an edge with it, and have two edges in common with the star pentagon, which is above the base. The other two edges are almost vertical, connecting the two levels.

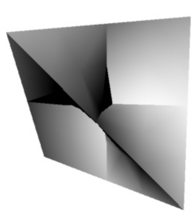

Figure 3: *The hemi-cube.*

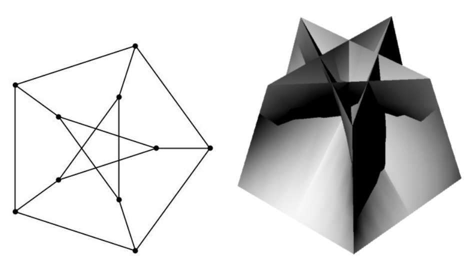

Figure 4: *The Petersen graph and an equilateral realisation of the hemi-dodecahedron.*

There is another realisation of the hemi-dodecahedron that is more symmetrical, having all its faces, which can be made equilateral, the same. It does not have the five-fold symmetry of Figure 4 but has tetrahedral symmetry instead. If triangular pyramids with angles of 36° are erected on the equilateral faces of the Jessen icosahedron that occurs in the jitterbug transformation of the octahedron the “inverted dodecahedron” results. In a similar way pyramids can be erected on the equilateral faces of the hemi-icosahedron to define the edges of this hemi-dodecahedron (Figure 5).

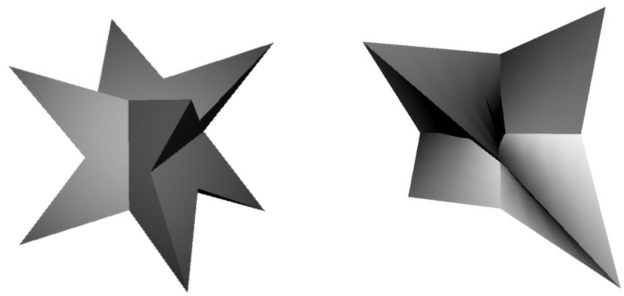

Figure 5: *The inverted dodecahedron and its hemi form.*

Another realisation can be constructed by bending the “ears” of the faces of the hemi-dodecahedron until the tips arrive at points antipodal to their starting positions. Again the resulting pentagons can be made equilateral (Figure 6). Figure 7 shows three views of the resulting polyhedron.

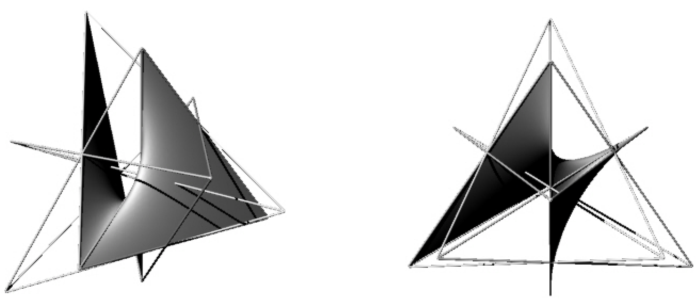

Figure 6: *Two views of an equilateral pentagon along with the edges of a hemi-dodecahedron.*

The tetrahemihexahedron has mirror symmetry, but the hemi-icosahedron derived from it exists as two enantiomorphic forms depending on which diagonals of the square faces are cut. In the same way the hemi-dodecahedra in Figures 7 and 9 seem to have mirror symmetry, but it is broken when the faces are identified. In fact the pentagonal faces are not mirror symmetrical as can be seen in Figure 6.

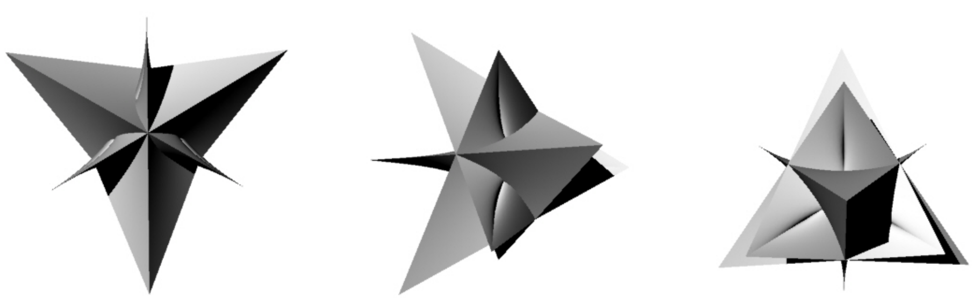

Figure 7: *Three views of another isohedral equilateral hemi-dodecahedron.*

Another interesting model with non-planar faces is the hemi truncated octahedron. It has four hexagonal faces, one of them planar, and three crossed quadrilaterals. Going around the regular hexagon its edges join alternately with the three non-planar hexagons and the quadrilaterals. On any of the other hexagons the adjacent edges join with diagonal edges of two quadrilaterals. The remaining three edges join with both the other hexagons and the other quadrilateral. Figure 8 shows a view of the polyhedron both with and without the regular hexagon.

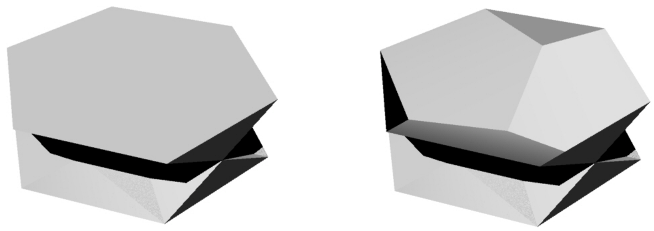

Figure 8: *The hemi truncated octahedron with and without the regular hexagon.*

Conclusion

Projective polyhedra have been considered from a theoretical point of view [2] but there have been few attempts to investigate possible physical realisations. Most famously Max Bill based sculptures on the Möbius strip but never thought about closing it to form the projective plane. Carlo Séquin considered possible artistic applications of smooth surfaces that are embeddings of the projective plane [5] but the possibilities for sculpture based on objects such as the equilateral isohedral hemi-dodecahedra remain to be explored.

Acknowledgements

Many of the examples described would not have been discovered without the use of Hedron [1], a program that generates VRML images of polyhedra from their abstract (combinatorial) definition, written by Jim McNeill. In some cases the image was suitable for use immediately but, because most of the examples do not have regular (or even planar) faces so the output is distorted, more work was needed and the final image was created using Rhino.

References

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