Horosphere, Cyclide and 3d Hyperbolic Tilings

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Abstract

We explore different ways to visualize three dimensional hyperbolic tilings using two dimensional cross sections

Introduction

Three dimensional tilings of the hyperbolic 3d-space (H^3) are difficult to visualize. In the standard models of H^3 like the Poincaré Ball (PB) or Upper Half Space (UHS) [5] the tiles become very small as they approach the boundary of H^3. The complete tiling viewed from a point outside the model is actually rather boring – just a round ball (PB) or half space (UHS). It is much more interesting to look at the tiling from an inside point. One way is to place the viewer in the virtual reality environment and simulate the view. The first inside view visualization was presented 25 years ago in the movie "Not Knot"[3]. More recently these ideas were resurrected [4] and ported into interactive virtual reality applications, which can run on smartphones[2]. Such inside-view visualization requires replacing solid tiles with some kind of a frame construction because the solid tiles are hard to view from the inside. In this paper, we are testing ideas of viewing H^3 tilings as the intersections of the tilings with various surfaces in H^3 – a hyperbolic plane, a hyperbolic sphere, a horosphere, a lens (an equidistant surface to a hyperbolic plane), and a cyclide (in our case, it is a surface of equal distance from a hyperbolic line). These surfaces can be naturally mapped onto the Euclidean plane for an easy 2d display. The view in such cross sections may highlight various properties of the tiling, such as symmetry and periodicity. Also, the cross section patterns have interesting aesthetic properties which makes them suitable for decorative purpose. We will be working with both the PB and UHS models as needed. As examples we will use a H^3 tiling generated by reflections in the sides of a compact hyperbolic tetrahedra with kaleidoscopic angles (Coxeter notation $[(3^3,4)]$ and $[(3^3,5)]$).

Hyperbolic Plane

The hyperbolic planes in the PB model are represented as the spheres or the planes orthogonal to the horizon (the surface of the Poincaré Ball). The simplest ones are the planes passing through the center of the ball. They are already flat. The intersection of the tiling with such a plane (Figure 1c) has a familiar appearance of a regular tiling of a 2d hyperbolic plane (Figure 1a). The pattern has several larger shapes near the center and their size decreases rapidly as it approaches the horizon of the plane (the boundary of the circle). In contrast to a 2d tiling, the shapes in the cross section do not look similar to each other because they are the cross sections of 3d tiles at various odd angles. However, the shapes are bound by the hyperbolic lines, which in this model are the circles orthogonal to the horizon.

The hyperbolic planes in the UHS model are the spheres or planes orthogonal to the horizon (the flat boundary of UHS). The simplest ones are the planes orthogonal to the horizon. The UHS plane cross section (Figure 1d) looks similar to the 2d tiling in the upper half plane model (Figure 1b). Shapes size rapidly decreases near the horizon. The shapes are bound by the hyperbolic lines, which are the circles or lines orthogonal to the horizon.

Lens

A lens is the equidistant surface to a hyperbolic plane. A lens in the PB model can be represented as a sphere or a plane which intersects the horizon sphere at at acute angle. Different angles correspond to different distance values. The simplest (flat) lens is a plane which does not pass through the center of the ball. The pattern in such a cross section (Figure 1e) behaves similarly to the hyperbolic plane cross sections – the shapes diminish in size near the horizon. However the rate of decrease varies and is controlled by the distance value of the lens. Another new property of such a pattern is the fact that the shapes' boundaries are not hyperbolic lines anymore but a more general mix of circles.

The lenses in the UHS model are spheres or planes intersecting the horizon plane at an acute angle. The simplest to represent are tilted planes (Figure 1f). Similarly to the PB model we have shapes of decreased size as they are approaching the horizon. The rate of decrease is controlled by the distance value of the lens (the angle of the plane).

Horosphere

A horospheres is the Euclidean plane embedded into H^3 . A horosphere can be represented in both models as the sphere tangent to the horizon. The most convenient horosphere model is the plane parallel to the horizon in the UHS model. It can be thought of as the tangent sphere of infinite radius with the tangent point at infinity. All horospheres are equivalent to each other, there is H^3 isometry which maps any horosphere into any other. Therefore, any horosphere is suitable for making a cross section. A horosphere can be thought of as the limit of lenses as the distance parameter tends to infinity.

The cross section with the horosphere (Figure 2a) has several striking properties: the pattern consists of the complex web of intersections of circles which has the same scale anywhere on the plane, it has similar uniform appearance but does not repeat itself. If the horosphere is properly aligned with the symmetry elements of the tiling the cross section pattern may have the corresponding symmetry but, in general, the symmetries of the pattern are only partially visible in the horosphere cross section in places where the horosphere intersects the location close to the vertices of the tiling.

Cyclide

A cyclide is an equidistant surface to a hyperbolic line. A cyclide has three independent parameters – two end points of the line and the distance value. In the UHS model, the hyperbolic lines are half circles or vertical lines orthogonal to the horizon. The simplest cyclide is a conic cyclide which is an equidistant surfaces to a vertical line (a hyperbolic line with one vertex at infinity). Such a cyclide is the infinite vertical circular cone with the apex at the horizon. The angle of the cone corresponds to the distance value. Other cyclides can be obtained from the conic cyclide via isometry of H^3. They looks like banana with two ends touching the horizon. To flatten the surface of the cyclide we first transform it into the conic cyclide using isometry of H^3 and unwrap the cone into the infinite sector by cutting it along straight line via its apex. The sector is mapped onto the infinite horizontal band using conformal mapping of the complex plane $log(z)$. This procedure maps the apex of the sector into the negative infinity and maps two sides of the sector into parallel horizontal lines. The result is shown at Figure 2b. The cyclide cross section pattern has the uniform scale everywhere similar to a horosphere cross section. However, the important difference of the cyclide cross section pattern is its periodicity. The pattern is always periodic in the vertical direction because y-axis is mapped into a polar angle of the point on the cone surface. In general, the pattern has no other period. However, if the end points are the limit points of the same H^3 isometry of the tiling the pattern has another period, as in case of Figure 2b.

Hyperbolic Sphere

A hyperbolic sphere can be most naturally represented in the PB model. Any sphere inside the ball represent a hyperbolic sphere in H^3 . The intersection pattern on such spheres will be non uniform in general. The pattern on one side of the sphere, which is closer to the horizon, will have smaller size than on the opposite side. The spheres centered at the ball's center will have a pattern of uniform size. The sphere is the only surface we used which cannot be flattened without distortion. It may be be confomally flattened with scale distortions using the stereographic projection or may be left as a sphere (Figure 2c). Most useful property of the sphere cross section is that it provides a truthful visual representation of a spherical sub symmetries of the tiling as in the case of Figure 2c which displays icosahedral sub symmetry of [(3^3,5)] tiling.

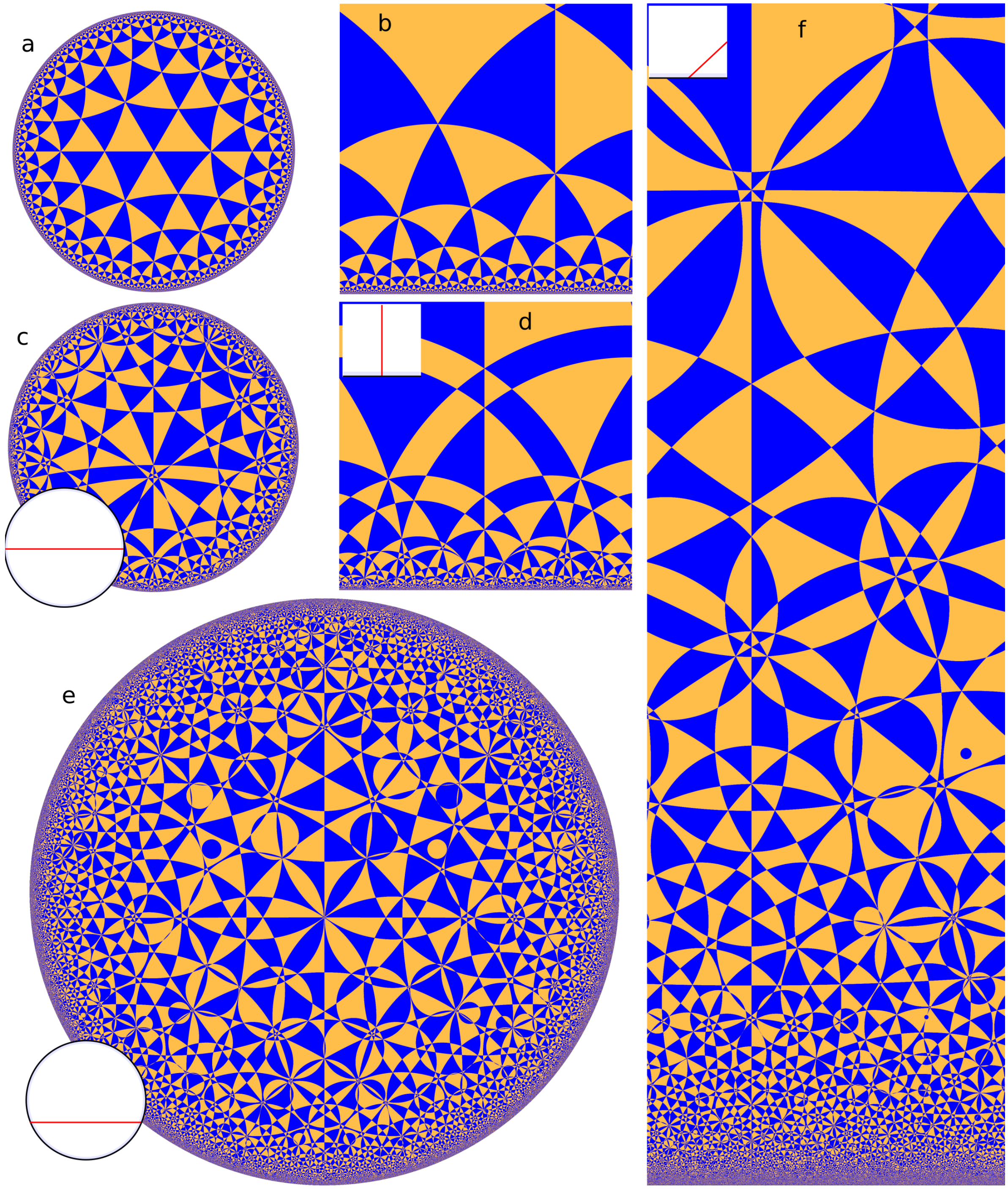

Figure 1: *The 2d triangular tiling (*334) in the circle model (a) and in the upper half plane model (b). Intersection of the tetrahedral tiling [(3^3,4)] with a plane in the PB model(c) and in the UHS model (d). Intersection of the tetrahedral tiling [(3^3,4)] with a lens in the PB model (e) and in the UHS model (f)*

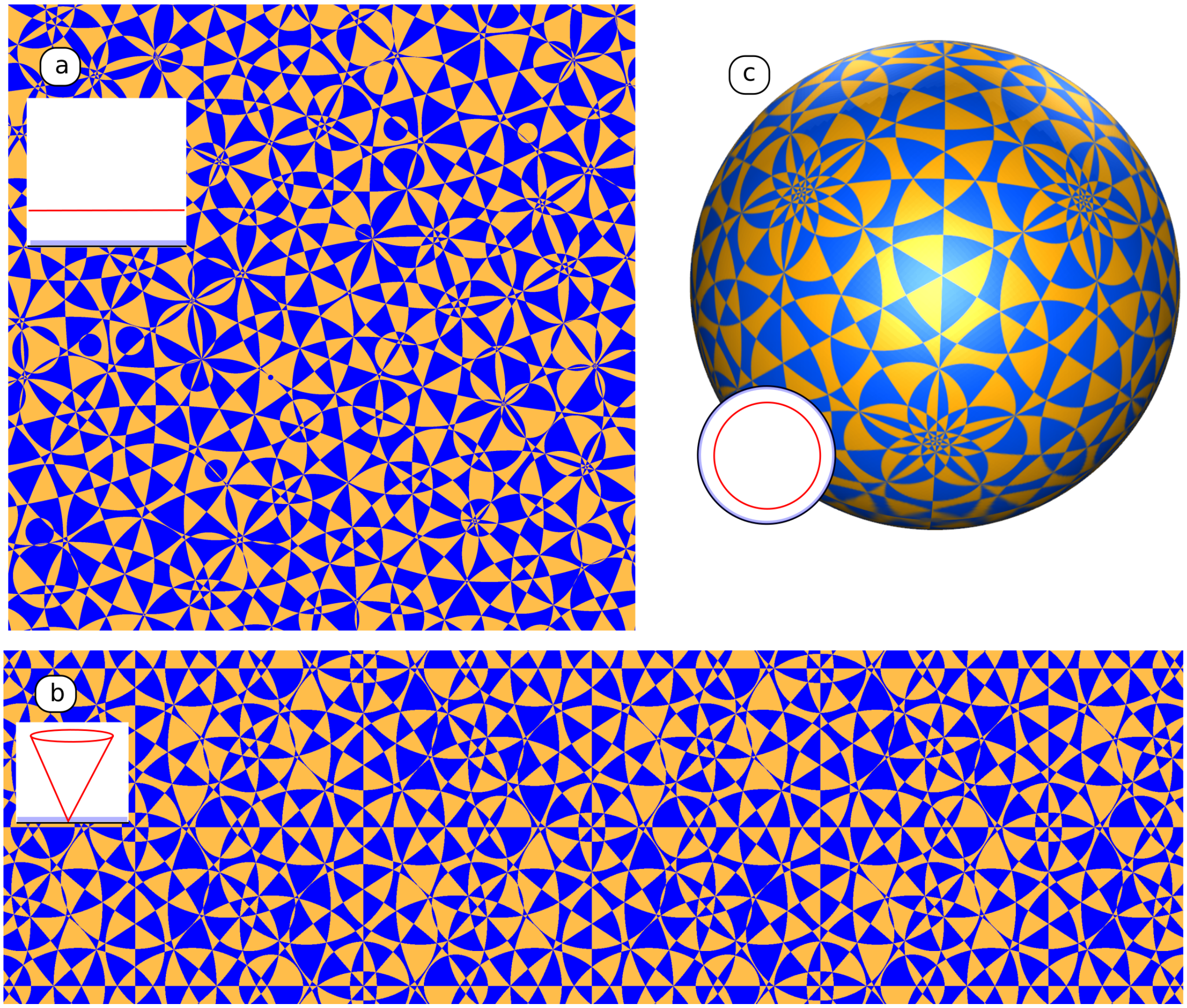

Figure 2: *Intersection of the tetrahedral tiling [(3^3,4)] with (a) a horosphere, (b) a cyclide. Intersection of the tetrahedral tiling [(3^3,5)] with a sphere (c).*

Conclusion

There are numerous tilings of H^3; therefore, interactive environment, where parameters can be changed and the feedback obtained instantly, provides the best opportunity for studying cross section patterns. We refer reader to the web page [1] with links to an interactive web application and videos of cross section animations.

References

[1] V. Bulatov. "Hyperbolic Cross Sections." http://bulatov.org/math/1807

[2] V. Hart, A. Hawksley, E. Matsumoto, and H. Segerman. "Non-Euclidean Virtual Reality I: Explorations H^3." *Bridges Conference Proceedings*, Waterloo, Canada, 2017, pp. 33–40

[3] C. Gunn and D. Maxwell. "Not Knot." video, dist. A.K. Peters, 1991.

[4] R. Nelson and H. Segerman. "Visualizing Hyperbolic Honeycombs." *Journal of Mathematics and the Arts,* vol.11, 2017, pp. 4–36.

[5] J. Ratcliffe. *Foundations of Hyperbolic Manifolds.* Springer, 2006.