Phoenix – Symbol of Mathematics

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Abstract

The phoenix is a mythical firebird that burns to death once in an eon and is reborn from its own ashes. Thus it is a symbol of resurrection and eternity. I used it as a symbol of mathematics in a profoundly mathematical mosaic with the same name. This article explains how the following mathematical concepts were applied to the opus: aperiodic tilings, certain minimal aperiodic tile sets and the einstein problem.

Introduction

What is mathematics? One half theorems, one half proofs and one half sheer imagination. It is extremely strict and definite, yet things like computers have literally been imagined into existence by mathematicians. During history, it has come across a number of interesting problems and a few mysteries. However, one topic has brought an end to mathematics as it was known at the time – repeatedly: infinity.

When Zeno first presented his four paradoxes, mathematics entered into a crisis. The ancient Greeks sought another way to talk about mathematics. It became effectively redefined, based on line segments rather than integers [1]. Later, in the twentieth century, the search for the foundations of mathematics threw it again into a crisis. The Incompleteness Theorem, the Undecidability Problem and the Continuum Hypothesis gave us a real sense that we do not really know what we are talking about. These too are closely related to the concept of infinity.

Yet, neither of these crises destroyed mathematics. If anything, it became more vibrant. On both occasions mathematics was born again, different, yet the very same, much like the phoenix. Mathematics is one of the most mysterious things in existence. Its radiant beauty is only visible to those who seek it.

The humble aim of this mosaic is to visualize that mystery. I approached the nature of mathematics via paradoxes:

- Strictly formal and highly creative – the phoenix is a symbol of creation, but has been embedded in a regular grid of squares.
- Visual representation of cutting edge mathematics and research with world records – the Socolar-Taylor monotile was only published in 2011 [5] and the Jeandel-Rao tile set put in the ArchiveX in 2015 [3].
- The finite and the infinite – the featured tilings are based on minimal finite tile sets that can cover the infinite plane with no finite period.
- The organized and the organic – the general layout putting the regular hexagon in juxtaposition with the freely placed tiles.
- The absolute and the personal – uncut standard-mosaic tiles were placed in a regular square grid by hand, a tribute to blackboard and chalk.
- The invisible and elegant beauty. In many common lighting conditions and without understanding of relevant mathematics, the opus seems rather boring and incomprehensible, but in spotlight – like direct sunlight – the phoenix motif jumps out, gleaming as if it was made of pure light. And when light clouds are passing the sun, the reflection seems to be alive with vivid colors.

Artistic Description

Phoenix is a mosaic embedded in plywood, 50 cm x 80 cm (20” x 32”), portrait (Figure 1).

In the upper part is a massive regular hexagon, filled with a regular square pattern tiling, tiles laid freehand. In the lower left, the hexagon is balanced by three irregularly placed tiles, like sparks flying from the tip of the bird’s wing.

The phoenix motif is drawn by a set of glass tiles with iridescent finish embedded in a field of matte ceramic tiles. Individual tiles are 20 mm squares with a 3 mm spacing. The colors of the two types of tiles are close matches, so while there are eight kinds of tiles, there are only four different colors, two kinds for each color.

The colors of the tiling form strips in the SW-NE direction, which is also the direction of the flight of the phoenix, giving it an air of ascent. The black, red, blue and green symbolize respectively ash, fire, air and life.

About Tilings

A brief introduction to tilings is necessary here [4]. A *tile set* is a finite set of *prototiles*. It implements local rules – e.g. different shapes – dictating how the individual tiles may be placed relative to each other. A *patch* is a finite area covered by tiles. A covering of the Euclidean plane by copies of prototiles without gaps or *overlays* and abiding by the local rule is a *valid tiling*. A tile set allowing a valid tiling is called a *valid set*. A valid tile set is *periodic* if it allows a patch to be copied infinitely and placed seamlessly next to itself throughout the plane without rotation or reflection of the patch. All other valid tile sets are *aperiodic*. The tiling problem asks how to determine if a given tile set is valid. The question is nontrivial and connected to the undecidability problem. Another important question is the still-open einstein problem: is there a valid aperiodic tile set with just one tile?

Jeandel-Rao Set

Wang tiles are unit squares with colored edges. Their local rule states that they may not be mirrored or rotated, and two tiles may only be placed next to each other if their adjacent edges have the same color. A common way to visualize Wang tiles is to divide the squares into right-angled triangles. The triangles are then colored with the edge color, in literature usually denoted by a number (Figure 2). A patch is presented as squares of the edge colors. It is customary to show the tiles as horizontal rows, so the edge squares become angled by 45 degrees (Figure 3).

The search for a minimal Wang tile set ended in 2015, when Emmanuel Jeandel and Michaël Rao showed that there are no aperiodic tile sets with fewer than 11 tiles, and that there exists an aperiodic set of Wang tiles with 11 tiles over 4 colors (Figure 2) [3]. (It was already known that no aperiodic sets with 3 colors or fewer exist [2].)

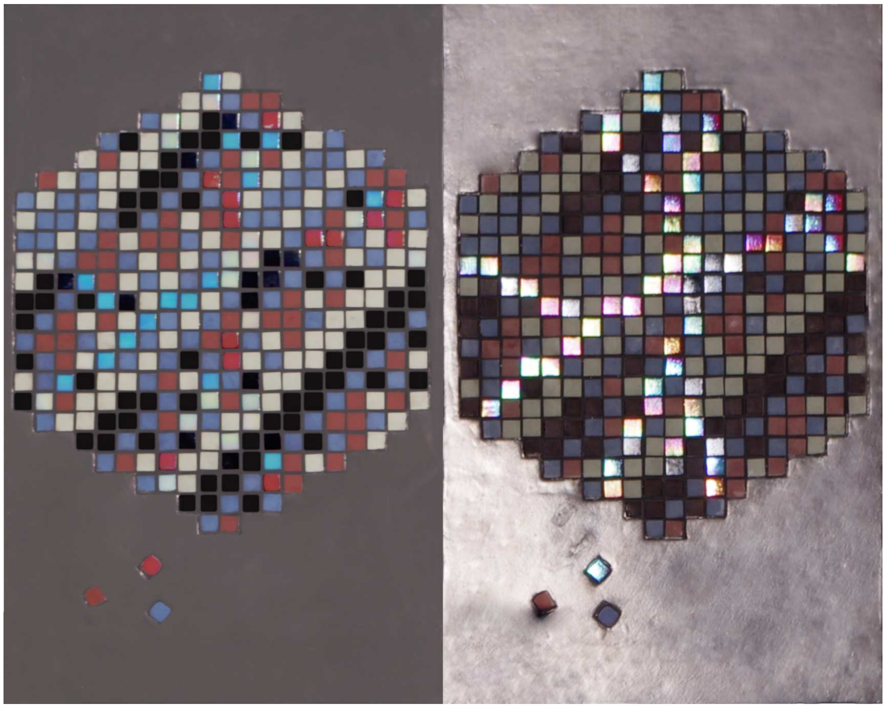

Figure 1: *The mosaic opus in diffuse and hard light (right).*

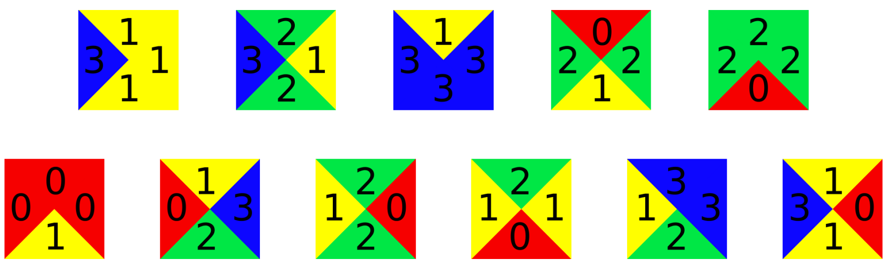

Figure 2: *Jeandel-Rao set.*

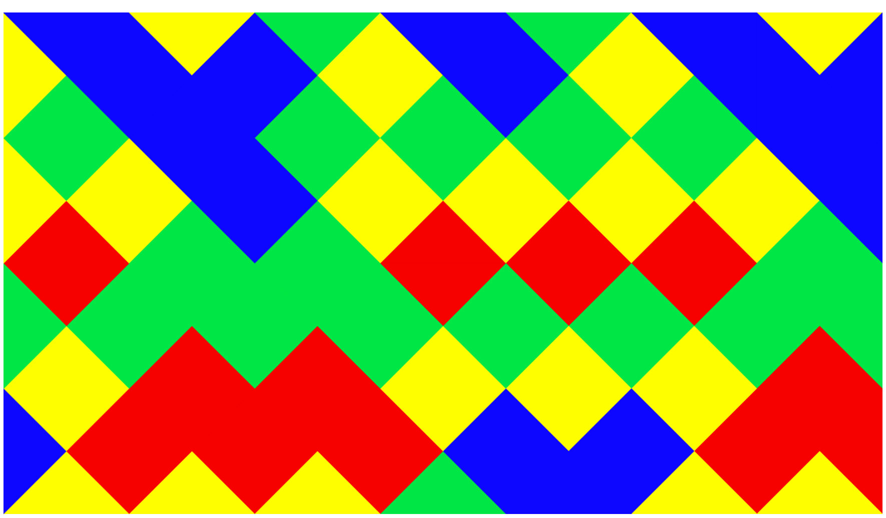

Figure 3: *A patch based on the Jeandel-Rao set.*

Socolar–Taylor Monotile

The Socolar–Taylor monotile provides a partial solution to the einstein problem. It is a single tile with a local rule that allows only aperiodic tilings, but the rule cannot be enforced in two dimensions by the tile shape alone. However, three-dimensional implementations of the monotile exist. The tile is a regular hexagon with a matching rule based on the decorations on its surface. Rotations and reflections of the tile are allowed [5].

To simplify the design process I have modified their original notations (Figure 4 a). In my version the matching rules are:

1. The black bars must be continuous across the edges and
2. The arrows heads must continue as arrow tails at the other end of the seam (Figure 4 b).

These rules together allow tiling of the entire plane, but only aperiocally. In addition, they are fully determined by certain points on the edges and corners of the tiles. The interior of the tile has no effect on the aperiodicity properties. The phoenix design (Figure 5) corresponds to the monotile diagram above with the marker pixels coding to the arrowheads and bars.

Designing the Opus

The general design principle was "mathematics first". While visualizing just one of these special tilings would have been a justification for a piece, it was too trivial to satisfy my artistic ambitions. From the start my main challenge was to figure out how to fit in both of them.

A special characteristic of the Socolar-Taylor monotile was its shape, which had to be reflected by the main mosaic area. To be practical the finished piece had to both be portable and have enough tiles to draw a meaningful motif. It was also necessary that the motif in the exact final patch could to be reflected and rotated to produce the other versions of the monotile. Thus the total mosaic area became a hexagon that fits in a 20x25 tile rectangle. This, and the decision to use the 8:5 approximation for the golden ratio, dictated the size of the frame.

I started to design the actual motif by marking the special tiles used to code the Socolar-Taylor conditions. The corner signals are binary, so a blank state could be used to encode the arrow tail:

1. If a hexagon has an iridescent tile at an edge, there has to be an iridescent tile next to it in the adjacent hexagon.
2. If a hexagon has an iridescent tile at a corner, then the corner at the other end of the seam departing from the tile must have a ceramic tile.

To design the theme, I needed locations for the marker pixels. The corner case was trivial. For the sides they could be chosen fairly freely, as long as I could replicate adjacent markers in the rotated and reflected versions, and I picked the ones three squares away from the corners. Under these conditions, I sketched the motif (Figure 5). In the early designs, I had a problem with overcrowding the patch with iridescent tiles, which made the motif a bit blurred when hard light hit it.

The media set requirements for another adaptation. The monotile is a regular hexagon, which I had to approximate in a square grid. To allow tiling, the opposite sides of the hexagon must have a matching profile in the square grid. Some artistic tweaking is needed, when this tiling is taken beyond the opus. The phoenix motif has to be redrawn onto the square grid for every orientation and its reflection.

The colors were dictated by the Jeandel-Rao set. Their paper [3] included a larger patch similar to the above example (Figure 3). For the phoenix design, I rotated that patch by 45 degrees counterclockwise and chose a suitable area from it. The tile colors were assigned according to my chosen palette. For the the tiles in the phoenix motif, I assigned numbers from 1 to 4, while the background was coded by letters from A to D. Finally, I assigned colors to the number–letter pairs. This produced a "map" to place the tiles (Figure 6).

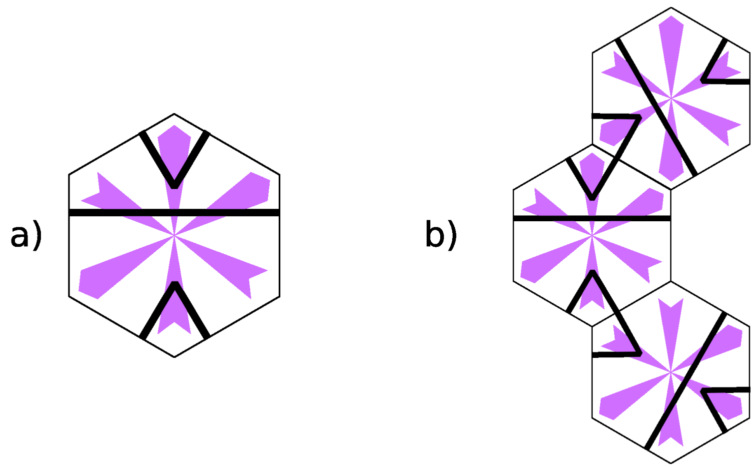

Figure 4: *a) The Socolar-Taylor monotile, adapted from the original article. b) The matching rules: The black bars are continuous. The purple arrowhead pointing down in the topmost tile fits the topmost arrow tail in the bottom tile, at the lower end of the vertical edge of the middle tile.*

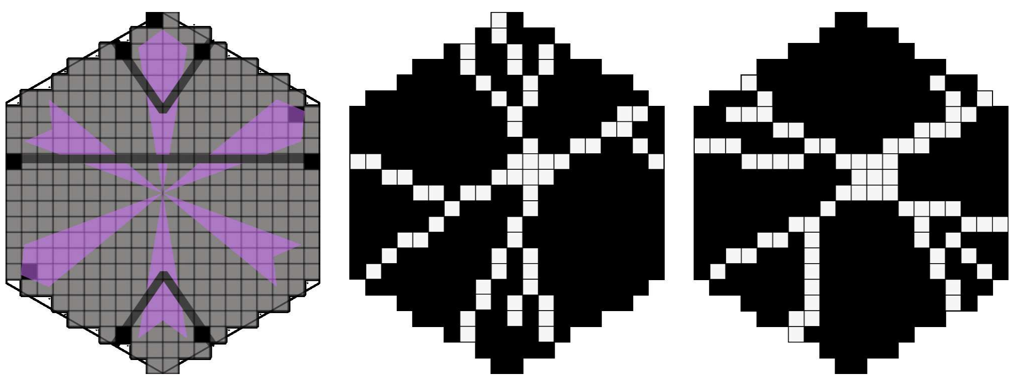

Figure 5: *Pixels coding the monotile arrows and bars; the phoenix design itself: the motif flipped horizontally and rotated one step counterclockwise (for demonstration only).*

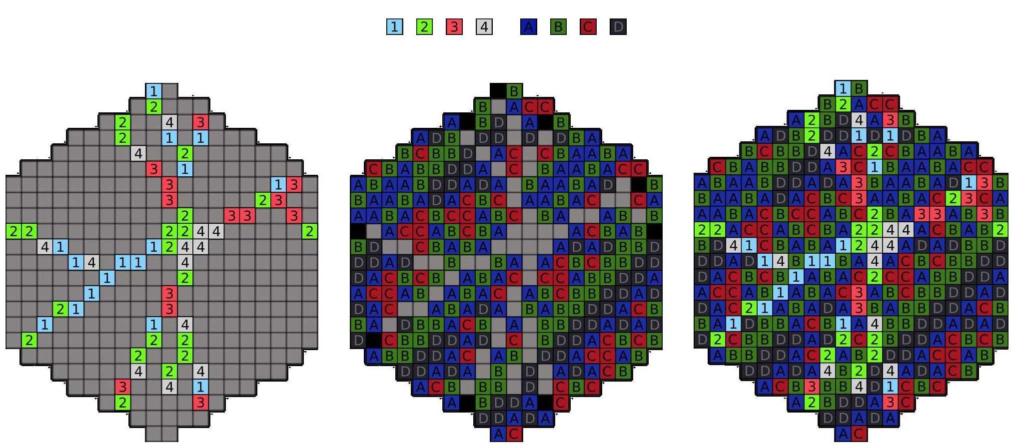

Figure 6: *The final phoenix design. Numbers stand for iridescent tiles, letters for matte. They are not related to the above tile set notation. 1/A: Blue, 2/B: Green, 3/C: Red 4/D: Black.*

Concluding Remarks

I made this mosaic specifically for the Bridges 2016 conference. Unfortunately, I could not write a paper on it then, because I started designing it only after the paper submission deadline. I got the general idea for it when I learned about the Jeandel-Rao set and Socolar-Taylor monotile in late 2015. They were were both cutting-edge mathematical research and both on their way to be landmark results in their field. Initially I set off with a simple artistic ambition: visualize them in a single opus. The connection to mathematics at large dawned only when the motif had occured to me.

Mosaics are generally unique and once set, cannot be fixed. However, the highly mathematical nature of this opus both allowed and required an exception. But as the frame deformed badly during the production, I took up the challenge of making a copy of the work. While I was crafting the copy, the original piece resettled and I found a mistake in it. I decided to break it to replace the tiles that were wrong, yielding two identical copies.

From the beginning the design process aimed further than just this piece. I endeavoured to develop a generic workflow for designing mosaic patterns using the principles from the theory of tilings and especially from the more challenging aperiodic sets. The Socolar-Taylor monotile presented a special challenge, because the rotations required adapted motif designs. The attempt was clearly a success. This process could be used in any pixel-based medium and even automated to create arbitrarily large aperiodic mosaic tilings. The the corner-edge notation (Figure 5) could also be easily applied in print or screen media to produce aperiodic figures without pixellation. A similar process may be applied to all tile sets. One just needs to work out how to implement the local rules and various tiles.

There is room for further refinement in at least three respects. First, the standard tile spacing is actually 2 mm, not 3 mm as in this opus. Second, when I was putting the piece together, I accidentally confused the order of the colors. This paper describes the actual outcome, rather than the initial design. The original design had 1/A coding red, 2/B black, 3/C blue and 4/D green. The third issue is the crudeness of the motif itself. As the aperiodicity only depends on the edges, the interior may be tweaked. For larger designs, one might use a larger base hexagon, perhaps with smaller tiles, allowing higher resolution. On the other hand, the mosaic tiles themselves can easily be split. This allows far more intricate designs, but breaks the design principle of full automation.

References

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[2] H. H. Chen, W. G. Hu, D. J. Lai, and S. S. Lin. "Nonemptiness problems of Wang tiles with three colors." *Theoretical Computer Science*, vol. 547, 2014, pp. 34–45.

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